Monetary Policy with Inelastic Asset Markets*

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Abstract

I develop a New Keynesian model to study the transmission of both conventional and unconventional monetary policy through financial markets. The model's two key features are (i) heterogeneous financial intermediaries with downwards-sloping asset demand curves, and (ii) households that face frictions in reallocating their savings across intermediaries. The central bank directly controls the risk-free rate, whereas the risk premium is determined by the distribution of intermediaries' wealth and the central bank's purchases of risky assets. Interest rate hikes reduce long-term risky asset values, redistributing wealth away from risk-tolerant intermediaries and increasing the risk premium. Central bank asset purchases can initially stimulate investment by reducing the risk premium, but asset prices may undershoot when those purchases are unwound. Optimal policy simultaneously uses both interest rate cuts and asset purchases to stabilize asset prices during downturns.

Keywords: Monetary Policy, Macro-Finance, Financial Intermediation, Slow-Moving Capital

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1 Introduction

Monetary policy transmits to the real economy by affecting not only the risk-free rate, but also a wide range of asset prices. Empirically, it is well-documented that interest rate cuts ("conventional monetary policy") compress credit spreads by affecting term premia (Hanson and Stein, 2015; Gertler and Karadi, 2015) and by reducing risk premia across several asset classes (Bernanke and Kuttner 2005; Bauer, Bernanke, and Milstein 2023). In recent years, with short-term nominal rates near zero, central banks have sought to use large-scale asset purchases (LSAPs, or "unconventional monetary policy") to directly influence premia on financial assets and provide additional stimulus (Krishnamurthy and Vissing-Jorgensen, 2011). The wide-ranging effect of monetary policy on financial markets raises several key questions. Why do conventional and unconventional monetary policy affect asset prices, and how do these effects transmit to the real economy? How should central banks manage interest rates and LSAPs to optimally stabilize output and inflation?

Workhorse representative-agent New Keynesian models usually have little to say about the transmission of monetary policy through financial markets. The main conceptual issue is that in such models, all assets are typically priced by the household's consumption-based stochastic discount factor (SDF). Hence, for a given path of consumption, asset demand is *perfectly elastic* (Gabaix and Koijen, 2021), so changes in asset quantities do not affect prices. The central bank is powerless to affect asset prices through unconventional monetary policy (Wallace, 1981). Moreover, to the extent that conventional interest rate policy does not have a large effect on the *volatility* of consumption, it usually has a limited effect on risk and term premia in workhorse models: monetary policy operates primarily through changes in the risk-free rate.

This paper develops a unified theory of how conventional and unconventional monetary policy transmit to the real economy through financial markets. The model features a fast-moving financial sector that prices assets in the short run, whereas the slow-moving household sector prices assets only in the long run, creating a wedge between the SDF in financial markets and the household's consumption-based SDF. Importantly, financial intermediaries' asset demand curves are downwards-sloping (i.e., inelastic), so that asset quantities matter for prices. Central bank asset purchases affect prices by changing the *supply* of risky assets, whereas changes in interest rates affect the financial sector's capitalization and therefore risky asset *demand*. The model's contribution is both positive and normative. On the positive side, I use the model to study the effects of shocks to monetary policy and find results in line with empirical evidence. On the normative side, I characterize the optimal joint conduct of interest rate policy and central bank balance sheet policy.

The backbone of the model is a New Keynesian economy with capital set in continuous time. There is a representative household that works, consumes, and saves, and firms employ capital and labor to produce goods for consumption and investment. In financial markets, there are two assets: risk-free, short-term bonds and risky, durable capital. The central bank sets the nominal rate and can engage in asset purchases by issuing bonds to purchase capital (or vice-versa), rebating all profits to the household. Nominal rigidities allow the central bank's nominal rate policy to have real effects, and portfolio adjustment frictions give rise to a role for central bank asset purchases.

The household invests its savings in heterogeneous financial intermediaries – there is a *bond fund* that invests only in risk-free bonds and several *mixed funds* that can invest in both risky capital and bonds on the household's behalf. The first departure from standard New Keynesian models is that the household faces a friction in reallocating its savings across funds (in the form of transaction costs). Household portfolio adjustment is therefore sluggish, as in the data: when the (risk-adjusted) returns on capital exceed those on bonds, the household *gradually* reallocates its savings into the market.[1] Hence, the household's portfolio flows determine asset prices in the long run.

In the short run, asset prices are determined by equilibrium in financial markets. The second departure from a standard New Keynesian model is that funds' demand for risky assets is inelastic: they are subject to *investment mandates* that make it costly to deviate from a target portfolio weight on capital. Hence, funds are willing to increase their portfolio weights on capital only if they earn an excess return over bonds. Investment mandates are meant to capture institutional or contracting frictions that limit intermediaries' ability to aggressively shift their portfolio allocations in response to asset price movements.[2] With inelastic asset demand, central bank asset purchases will move prices, since they change the supply of risky assets. After laying out these key ingredients of the model, I log-linearize it to provide a sharp analytical characterization of equilibrium. Importantly, to first, order, the model's heterogeneous mixed funds aggregate to a single representative risk-bearing fund, henceforth referred to as "the market."

The model generates a novel mechanism through which conventional and unconventional monetary policy affects asset prices, which I term the *inelastic markets channel.* Three forces determine the excess returns (risk premium) on capital in financial markets: (1) the market's share of total wealth (its *risk-bearing capacity*), (2) the elasticity of the market's demand for

[1]See also Reis (2006) for a model of optimal inattention in which the household must pay a cost to collect information about the stock market. Brunnermeier and Nagel (2008) empirically document slow household portfolio adjustment.

[2]See Gabaix and Koijen (2021), who argue that in the data, mutual funds and other intermediaries tend to hold very stable portfolio shares of equity and bonds.

capital, and (3) the quantity of assets held by the central bank. The market's wealth share is the key state variable that determines the *level* of risky asset demand (since funds target a benchmark portfolio weight on capital), whereas the elasticity determines the *sensitivity* of the risk premium to central bank asset purchases. A shock that lowers risky asset prices, such as a decrease in productivity or an interest rate hike, decreases the market's risk-bearing capacity and increases the risk premium on capital. The central bank can neutralize these fluctuations in asset prices by purchasing capital, reducing the supply that the market has to absorb. In the long run, though, the household will gradually reallocate its savings into the the market, increasing risky asset demand. Hence, central bank asset purchases cannot affect the premium on capital in the long run (just as interest rate policy cannot affect real rates in the long run).

The inelastic markets channel has implications for the real economy as well. Whereas the short-term real interest rate determines the household's consumption demand (via a typical Euler equation), investment demand is determined by the overall level of asset prices, which depends on both the risk-free rate and the risk premium. Absent central bank asset purchases, a shock that lowers the market's risk-bearing capacity leads to a contraction in investment and output as well as deflationary pressure. The recession is prolonged by a persistent decline in the capital stock, which depresses the economy's productive capacity. Using this characterization of the inelastic markets channel, I study its implications for the transmission of shocks to both conventional and unconventional monetary policy.

The inelastic markets channel amplifies the effects of conventional monetary policy. An unanticipated interest rate hike raises the real rate, but it also decreases the market's risk-bearing capacity by lowering asset prices. This redistribution of wealth results in an increased risk premium: since the market's capitalization is lower, it requires a higher risk premium to hold the same quantity of capital. In turn, the increased risk premium lowers asset prices further, amplifying the initial shock and causing a larger decline in investment than in a typical New Keynesian model. This amplification of monetary policy is in line with empirical evidence: in the data, just as in the model, unanticipated interest rate shocks cause risk premia to move in the same direction (Kashyap and Stein, 2023).

An unwinding of previous central bank asset purchases can cause asset prices to undershoot. I study an "asset purchase shock" in which the central bank purchases a large quantity of assets and then gradually sells them back to the market. The initial asset purchase lowers the risk premium, making investment in the market unattractive from the household's perspective. The household therefore reallocates savings towards bonds, reducing the market's risk-bearing capacity even as risk premia remain low. When the central bank unwinds its initial asset purchase, then, the market requires a higher risk premium to buy those assets back. Asset prices fall below their steady-state level, causing a reduction in investment demand. Long-lasting asset purchases are especially likely to eventually lead to recessions when unwound. The model thus demonstrates that a reduction in the size of the central bank's balance sheet does not merely undo previous stimulus and return asset prices to normal levels – it may be so contractionary that it causes a recession.

I then turn to optimal policy. As a benchmark result, I show that there is no "divine coincidence": even if the central bank manages interest rates to close the output gap, it does not necessarily achieve the first-best. This is because changes in interest rates affect asset prices and therefore the market's risk-bearing capacity, which in turn leads to fluctuations in risk premia and the investment rate. I show that a competitive equilibrium that achieves the first-best is characterized by two features: the central bank must close the output gap and ensure that the ratio of asset prices to output (the *price-output ratio*) is equal to a given target. The latter condition ensures that the investment-consumption ratio is optimal: if the price-output ratio is too low, for instance, there is underinvestment. With asset purchases, the central bank can achieve the target level of the price-output ratio and use interest rate policy purely to close the output gap. Hence, to make the central bank's problem non-trivial, I assume that the central bank must pay a cost to hold risky assets, which could reflect a disadvantage in administering loans or political constraints. Then, it is no longer possible to costlessly achieve the first-best.

I take a Ramsey approach to the central bank's policy problem and find optimal second-best allocations subject to implementability constraints. For tractability, I take a second-order approximation to the welfare function. From a methological standpoint, given that some of the implementability constraints are forward-looking, I use the recursive contracting approach outlined in Marcet and Marimon (2019). I first ask how interest rate policy and asset purchases should be used in conjunction with one another and then characterize the optimal path of asset purchases, given an initial shock to the market's risk-bearing capacity.

Interest rate cuts and asset purchases are complements. Whenever the central bank purchases assets, it also cuts rates to run the economy hotter (i.e., it raises the output gap). The intuition is straightforward. The central bank purchases assets when risky asset prices are too low. Just like asset purchases, interest rate cuts can be used to indirectly reduce risk premia by boosting the market's risk-bearing capacity, so whenever the central bank finds it optimal to purchase assets (which is costly), it also cuts interest rates at the cost of overheating the economy.

Following a negative shock, the central bank commits to a prolonged asset purchase. It pursues a "late-exit" strategy and maintains assets on its balance sheet even after financial markets have recovered. The central bank's commitment not to unwind its balance sheet too fast serves to make asset prices more sensitive to a given balance sheet expansion. This policy causes asset prices to remain too high for some time after markets have recovered, but it permits the central bank to economize on the costs of holding large quantities of assets. If the market believed the central bank would immediately sell assets back once its risk-bearing capacity had recovered, the initial announcement of the purchase would not be as effective in increasing asset prices. The model can therefore rationalize why some central banks have committed to maintain sizeable balance sheets instead of winding them down once economic conditions normalize.

The rest of the paper is organized as follows. The remainder of this section discusses the related literature. Section 2 presents the model and defines an equilibrium. Section 3 provides a log-linear approximation to the model's equations and an analytical characterization of the equilibrium. Section 4 studies the economy's response to monetary shocks. I derive the optimal policy in Section 5. Section 6 concludes. All proofs can be found in the Appendix.

Related Literature: My model builds on three strands of the literature. The first is the extensive macro-finance literature that incorporates limits to arbitrage and intermediaries with limited risk-bearing capacity. Methodologically, Gabaix and Koijen (2021) is closest to my paper: the authors present a macroeconomic model in which intermediaries are willing to adjust their portfolio weight on stocks only if risk premia are high enough, generating asset pricing implications similar to those in my model. That paper is more focused on studying the effects of exogenous flows across funds, whereas the focus of my work is on understanding the implications of inelastic intermediary asset demand for monetary policy. Gabaix and Maggiori (2015) and Itskhoki and Mukhin (2022) develop models of exchange rate determination in which intermediaries have a limited capacity to bear currency risk – as in my paper, Itskhoki and Mukhin (2022) demonstrate that interest rate policy and asset market interventions must act in concert to achieve the first-best. Ray (2019) and Dordal i Carreras and Lee (2022) develop preferred-habitat models of the term structure of interest rates to study the effects of yield curve control policies.

My paper is also related to the literature that studies frictions in household portfolio adjustment. Duffie and Sun (1990), Duffie (2010), and Abel, Eberly, and Panageas (2013) study household portfolio allocation in the presence of transaction and information collection costs. Chien, Cole, and Lustig (2011, 2012) develop a macroeconomic model in which some households allocate a fixed fraction of their portfolios to stocks, provide a tractable characterization of equilibrium, and study the model's implications for asset pricing. Bacchetta, Davenport, and van Wincoop (2022) add portfolio adjustment costs to an international model and derive implications for exchange rate dynamics and international portfolio flows.

Finally, my paper is connected to the emerging literature on the relationship between monetary policy and risk premia. Brunnermeier and Sannikov (2016) study an Aiyagari-style economy with financially constrained intermediaries in which monetary policy and asset purchases can affect *idiosyncratic* risk premia. Silva (2022) constructs a model in which central bank asset purchases can affect risk premia by transferring risk from marginal to inframarginal ("passive") investors. Whereas that paper's quantitative model is solved nonlinearly using numerical methods, I take an approximation to my model's equilibrium to provide analytical results. Kekre and Lenel (2022) build a quantitative New Keynesian model with heterogeneous agents and demonstrate that the redistributive effects of interest rate shocks can account for the effects of monetary policy on risk premia. My model differs in that it studies the *joint* conduct of conventional and unconventional monetary policies.

2 Model

I consider a New Keynesian economy with capital set in continuous time $t \in [0, \infty)$. There is a representative household that works, consumes, and saves. The household faces a cost to adjust its portfolio (which ensures that open-market operations are non-neutral). As in most New Keynesian models, there are monopolistic retailers with sticky prices that hire labor to produce differentiated varieties of intermediate goods. A representative firm rents capital and aggregates intermediate goods to produce final output that can be consumed or invested. Productivity shocks are the only source of aggregate uncertainty.

There are two funds that trade capital and short-term real bonds in competitive financial markets: a bond fund $i = 0$ (that is constrained to hold only bonds) and I mixed funds $i \in \{1, \ldots, I\}$. Funds are owned by the household and maximize their returns with respect to its stochastic discount factor. However, each mixed fund must pay a cost to deviate from a pre-specified target portfolio weight on capital (its *investment mandate*).

The central bank has two tools: it sets the short-term nominal interest rate and can engage in direct asset purchases, issuing short-term bonds to buy risky capital (or vice-versa). It transfers all profits and losses back to the household.

2.1 Supply side

Technology: Productivity A_t evolves according to a geometric Brownian motion,

$$\frac{dA_t}{A_t} = gdt + \sigma dZ_t.$$

Monopolistic retailers $j \in [0, 1]$ hire labor ℓ_{jt} to produce $X_{jt} = A_t \ell_{jt}$ units of intermediate good j. The final goods producer aggregates differentiated intermediate varieties X_{jt} into a composite intermediate good X_t, and then combines capital and the intermediate good to produce final output using a Cobb-Douglas technology:

$$Y_t = K_t^\alpha X_t^{1-\alpha}, \quad \text{where } X_t = \left(\int_0^1 X_{jt}^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}. \tag{1}$$

Here, $\alpha \in (0,1)$ denotes capital's share of output and $\epsilon > 1$ is the elasticity of substitution across intermediate varieties j.

Capital K_t depreciates at rate δ, and investment is subject to adjustment costs: an investment of $x_t K_t$ goods yields $\phi(x_t) K_t$ units of new capital, where $\phi(\cdot)$ is an increasing, differentiable, and concave function. For analytical convenience, I take

$$\phi(x) = \nu \log x$$

for some constant $\nu > 0$, but this assumption is not essential for most of the main results. The aggregate resource constraint is then

$$Y_t = C_t + x_t K_t, \tag{2}$$

(with the obvious notation) and the capital stock evolves according to

$$dK_t = \big(\phi(x_t) - \delta\big) K_t dt. \tag{3}$$

Representative firm's problem: The representative final goods producer rents capital and purchases intermediate goods to produce final output, taking the nominal prices P_{jt} of intermediates and the rental rate of capital Ren_t^k as given.[3] Final goods sell in competitive markets at nominal price P_t. The representative firm can also invest final goods to produce capital, selling claims on new capital directly to funds at a nominal price $P_t q_t$, where q_t is the real price of capital. The firm's problem is

$$\max_{K_t, X_{jt}, x_t} \; P_t Y_t - Ren_t^k K_t - \int_0^1 P_{jt} X_{jt} dj + P_t \big(q_t \phi(x_t) - x_t\big) K_t \;\; \text{s.t.} \;\; (1).$$

The first-order conditions are standard. In particular, the rental rate of capital is equal

[3]The non-standard notation Ren_t^k is used to avoid confusion with the return on capital, which will be denoted dR_t^k.

to its marginal product plus the profits from investment:

$$Ren_t^k = P_t\left(\frac{\alpha Y_t}{K_t} + q_t\phi(x_t) - x_t\right). \tag{4}$$

Optimal investment is given by Tobin's Q:

$$1 = q_t\phi'(x_t) \quad \Rightarrow \quad x_t K_t = \nu q_t K_t. \tag{5}$$

Aggregate investment is therefore equal to a constant fraction of the market value of the capital stock $q_t K_t$. This is the channel through which changes in risky asset prices transmit to aggregate demand.

New Keynesian Phillips curve: Monopolistic retailers set their prices subject to quadratic adjustment costs as in Rotemberg (1982), taking nominal wages W_t and the demand curve for their goods as given. The representative final goods producer chooses the quantity of goods to purchase from each retailer to maximize profits, generating a downward-sloping demand curve for each retailer's variety. I defer the details of monopolistic retailers' problem to the Appendix, but I summarize the main result here. Monopolistic retailers' pricing decisions generate a standard New Keynesian Phillips curve:

$$\pi_t = \kappa\, \mathbb{E}_t\left[\int_0^\infty e^{-\rho s}\Lambda_{t+s}Y_{t+s}\left(\frac{W_t}{A_t P_t^X} - \frac{\epsilon - 1}{\epsilon}\right)ds\right], \tag{6}$$

where $\kappa > 0$ is a constant, Λ_{t+s} represents the household's marginal utility of consumption at time $t+s$, and P_t^X is the usual CES price index for intermediate goods. Retailers increase their prices when their actual markups $\frac{A_t P_t^X}{W_t}$ over marginal costs are below their desired mark-ups $\frac{\epsilon}{\epsilon - 1}$.

2.2 The household

The household has standard log-linear preferences over consumption and labor. Its lifetime utility is given by

$$\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\Big(\log C_t - \ell_t\Big)dt\right], \tag{7}$$

where $\rho > 0$ is the household's discount rate.

At time t, the household enters with savings S_{it} invested in each fund i and chooses flows $F_{it}dt$ into each fund. One unit of savings invested in fund i pays out a dividend $D_{it}dt$ at time t and earns a return dR_{it} from t to $t + dt$. I conjecture that the return on each fund's portfolio follows an Ito process:

$$dR_{it} = r_{it}dt + \sigma_{it}dZ_t.$$

The determination of each fund's returns and dividends is discussed in the next section. Hence, the household's savings in fund i evolve according to

$$dS_{it} = S_{it}(dR_{it} - D_{it}dt) + F_{it}dt. \tag{8}$$

The first substantive departure from a standard New Keynesian model is that the household faces frictions in adjusting its portfolio. When the household chooses flows F_{it} into fund i, it pays a quadratic transaction cost of $\frac{1}{2}\frac{\chi_i}{A_t}F_{it}^2$ goods, where χ_i is a parameter indexing the cost of portfolio adjustment into fund i.[4] Such transaction costs are typical in the literature on slow-moving capital (see Duffie 2010 or Gârleanu and Pedersen 2016), but generally these costs can be thought of as a stand-in for other frictions that cause sluggishness in household portfolio adjustment, such as inattention or information costs (see e.g. Reis 2006). In contrast to its investments in mixed funds, the household can frictionlessly adjust bond holdings B_t at time t. Bonds pay a real return $r_t dt$ from t to $t + dt$.

The household chooses consumption C_t, labor supply ℓ_t, flows F_{it} into each fund, and the change in its bond holdings dB_t to maximize (7) subject to (8) and the budget constraint

$$dB_t = \Big(\big(\frac{(1+\tau^\ell)W_t}{P_t}\ell_t + r_t B_t + \sum_{i=1}^{I} S_{it}D_{it}dt \big) - \big(C_t + \sum_{i=1}^{I}(F_{it} + \frac{1}{2}\frac{\chi}{A_t}F_{it}^2) \big) \Big) dt + dT_t, \tag{9}$$

where dT_t denotes transfers received by the household and $1 + \tau^\ell$ is a wage subsidy that ensures an efficient labor supply along the balanced growth path.[5] I assume that aggregate price and portfolio adjustment costs are rebated to the household, so that transfers consist of adjustment costs plus remittances from the central bank (described later).

The solution to the household's problem is summarized in the following proposition.

Proposition 1. *The household's optimal consumption path is characterized by the Euler equation*

$$r_t = \rho + \mu_{Ct} - \sigma_{Ct}^2, \tag{10}$$

where μ_{Ct} and σ_{Ct} are the drift and volatility of the household's consumption, respectively.

[4] The cost of portfolio reallocation is assumed to scale with A_t to ensure the existence of a balanced growth path.

[5] This type of subsidy is typical in Keynesian models, since monopolistic retailers' market power would otherwise imply inefficiently low labor supply in steady state. Further details are in the Appendix.

The household's choice of flows into fund i, F_{it}, satisfies

$$F_{it} = \frac{A_t}{\chi_i}\mathbb{E}_t\left[\int_0^\infty e^{-\rho s}\frac{C_t}{C_{t+s}}\frac{X_{i,t+s}}{X_{it}}\big(r_{i,t+s} - r_{t+s} - \sigma_{C,t+s}\sigma_{i,t+s}\big)ds\right], \tag{11}$$

where X_{it} is the quantity of savings remaining at time t per unit invested in fund i at $t = 0$, i.e., $X_{i0} = 1$ and $\frac{dX_{it}}{X_{it}} = dR_{it} - D_{it}dt$.

Since the household can frictionlessly adjust its bond holdings, it effectively discounts consumption at the risk-free nominal rate, yielding the standard Euler equation (10). A corresponding Euler equation does not hold for the returns on funds' portfolios. Instead, (11) demonstrates the household gradually increases its savings in fund i when it expects to earn high risk-adjusted *excess returns* $r_{it} - r_t - \sigma_{Ct}\sigma_{it}$ in the future. I discuss the solution to the household's problem in greater detail in Section 3, where I log-linearize the model.

2.3 Asset markets

Funds and the central bank trade two assets in competitive markets: claims on capital, which trade at real price q_t, and short-term real bonds, which are in zero net supply and pay an interest rate r_t. I conjecture that the price of capital follows an Ito process,

$$\frac{dq_t}{q_t} = \mu_{qt}dt + \sigma_{qt}dZ_t$$

The dividend paid by capital at t is the real rental rate $\frac{Ren_t^k}{P_t}$ given in (4). The real return on capital at time t is then

$$dR_t^k = \left(\frac{\alpha Y_t - x_t K_t}{q_t K_t} + \phi(x_t) - \delta + \mu_{qt}\right)dt + \sigma_{qt}dZ_t. \tag{12}$$

I denote fund i's capital holdings at t by K_{it} and the central bank's capital holdings by K_t^{CB}. The market clearing condition states that funds' aggregate capital holdings plus the central bank's must equal the aggregate capital stock:

$$\sum_{i=1}^{I} K_{it} + K_t^{CB} = K_t. \tag{13}$$

The bond market clears automatically by Walras' Law. Figure 1 provides a graphical representation of market participants' balance sheets and the flow of funds.

At time t, the fund i enters with net worth N_{it} and chooses its portfolio weight ω_{it} on

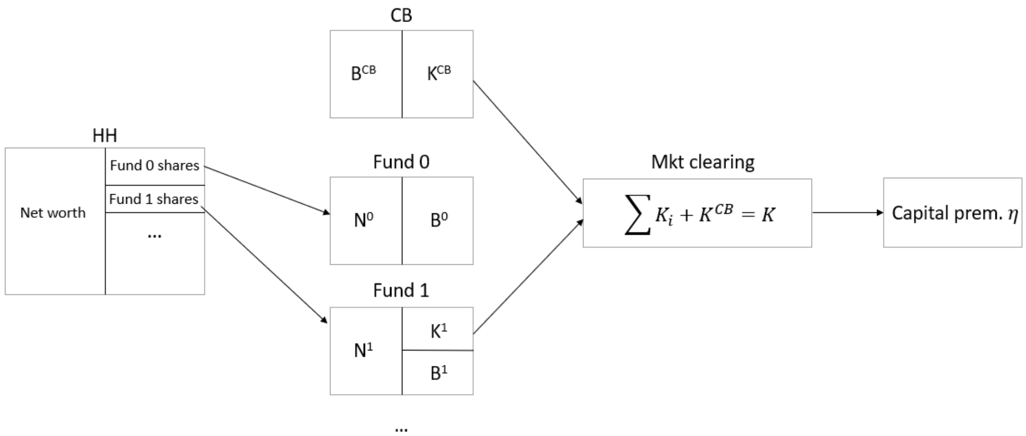

Figure 1: Illustration of financial markets in the model.

capital, with the remaining fraction $1-\omega_{it}$ invested in bonds. The second point of departure from a standard New Keynesian model is that funds are subject to *investment mandates* that make it costly for fund i to deviate from a target portfolio weight ω_i^* on capital.[6] When fund i invests a fraction ω_{it} of its portfolio in capital, it pays a cost of $\xi_i(\omega_{it}-\omega_i^*)N_{it}dt$ goods, where ξ_i is a convex and twice differentiable function with a minimum $\xi_i(0)=0$ (representing fund i's mandate).[7] The real return on the market's portfolio from t to $t+dt$ is

$$dR_{it} = (1-\omega_{it})r_t dt + \omega_{it} dR_t^k - \xi_i(\omega_{it}-\omega_i^*)dt. \tag{14}$$

Funds' mandates should be thought of as representing contractual constraints, institutional frictions, or rules of thumb that limit financial intermediaries' ability to shift their portfolio composition away from some benchmark allocation.[8]

At the beginning of each instant t, each fund pays out an exogenous fraction of its net worth $\gamma_t dt$, where the dividend payout ratio $\gamma_t = r_t$ is set equal to the real rate of return on bonds for simplicity.[9] The dividend paid to the household by fund i per unit invested is then $D_{it}=\gamma_t$. Each fund also receives inflows $F_{it}dt$ from the household. Fund i's net worth then

[6] Since bonds are in zero net supply, funds must be levered on average. Otherwise, the bond fund would have no one to lend to.

[7] This cost is rebated to the household as a transfer and therefore does not enter the resource constraint.

[8] Examples of intermediaries facing such constraints are mutual funds that target a particular stock-to-bond ratio, hedge funds whose managers' compensation depends on performance relative to a benchmark, pension funds that must guarantee a certain rate of return, and broker-dealers facing value-at-risk constraints.

[9] The assumption that γ_t takes this form is analytically convenient but inessential for the main results. The results would be largely the same with a constant payout ratio γ.

evolves according to

$$dN_{it} = N_{it}\big((r_{it} - \gamma_t)dt + \sigma_{qt}dZ_t\big) + F_{it}dt \tag{15}$$

Fund i's objective is to maximize the present value of dividends (using the household's stochastic discount factor), taking as given the returns on capital and nominal bonds, inflows from the household, and the household's marginal utility of consumption $\Lambda_t = \frac{1}{C_t}$:

$$\max_{\omega_{it}} \ \mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\Lambda_t\gamma_t N_{it}dt\right] \ \text{ s.t. } \ (14),\ (15),\ N_0 \text{ given.} \tag{16}$$

The solution to fund i's portfolio allocation problem is characterized in the following proposition.

Proposition 2. *Fund i's optimal portfolio weight on capital satisfies*

$$\xi_i'(\omega_{it} - \omega_i^*) = r_t^k - r_t - \sigma_{qt}\sigma_{Ct} + \sigma_{qt}\sigma_{vit}, \tag{17}$$

where v_{it} denotes the present value of a unit of savings in the fund i,[10] *and σ_{vit} is its volatility.*

Funds' optimality condition (17) demonstrates that, all else equal, a fund invests a greater share of its portfolio in capital when expected (risk-adjusted) excess returns $r_t^k - r_t - \sigma_{qt}\sigma_{Ct}$ are high. However, it does not necessarily adjust its portfolio weight so much as to completely eliminate the risk-adjusted excess return (as it would in a standard model). I return to funds' problem when discussing the equilibrium in Section 3.

2.4 Central bank policy

The central bank has two policy instruments: the nominal interest rate i_t and its capital holdings K_t^{CB}, which are financed by issuing a real quantity of bonds $q_tK_t^{CB}$. The nominal rate is related to the real rate by the Fisher equation,

$$r_t = i_t - \pi_t. \tag{18}$$

The (possibly stochastic) processes $\{i_t, K_t^{CB}\}$ are announced at $t = 0$. The central bank remits all profits and losses as transfers to the household period-by-period. For now, I do not specify the policy rules further. In Section 4, to derive positive results on the effects of (conventional and unconventional) monetary policy shocks, I assume that the central bank follows mechanical rules based on the current state of the economy. In Section 5, I study the optimal policy with commitment.

[10] The formula for v_{it} is in the Appendix.

2.5 Equilibrium

The equilibrium definition is standard. All agents and firms optimize, and prices adjust so that markets clear.

Definition 1. *An* ***equilibrium*** *consists of processes for aggregates* $\{Y_t, C_t, K_t, X_{jt}, \ell_t, x_t\}$*; prices* $\{P_t, P_{jt}, W_t, Ren_t^k, q_t, r_t, \pi_t\}$*; fund variables* $\{N_{it}, \omega_{it}, F_{it}, D_{it}, R_{it}, B_t\}$*; central bank policies* $\{i_t, K_t^{CB}\}$*; and cumulative transfers* $\{T_t\}$ *such that*

1. *Capital, intermediate inputs, and investment* $\{K_t, X_{jt}, x_t\}$ *solve the firm's problem, taking* $\{P_t, P_{jt}, Ren_t^k, q_t\}$ *as given;*

2. *Consumption* C_t*, flows* F_{it}*, labor supply* ℓ_t*, and bond holdings* B_t *solve the household's problem, taking* $\{W_t, r_t, D_{it}, R_{it}, T_t\}$ *as given;*

3. *Funds' portfolio choices* ω_{it} *are optimal, taking* $\{Ren_t^k, q_t, r_t\}$ *as given, and their returns, dividends, and net worth* $\{R_{it}, D_{it}, N_{it}\}$ *are consistent with their portfolio choices and inflows* $\{\omega_{it}, F_{it}\}$*;*

4. *Inflation* $\{\pi_t\}$ *obeys the Phillips curve;*

5. *Interest rates and the central bank's capital holdings* $\{i_t, K_t^{CB}\}$ *are consistent with the central bank's policy rule, and the real rate* r_t *satisfies the Fisher equation;*

6. *Transfers* $\{T_t\}$ *are equal to the central bank's profits plus portfolio and price adjustment costs;*

7. *All markets clear.*

3 The balanced growth path and log-linearization

In this section, I derive a *balanced growth path* (BGP) equilibrium with deterministic growth in which consumption, output, and asset prices grow at a constant rate. Then, I log-linearize around a BGP in order to characterize the economy's adjustment back to the BGP as well as its response to unanticipated shocks.

3.1 Balanced growth path equilibrium

In the deterministic economy, productivity is assumed to grow deterministically at rate g, $\frac{dA_t}{A_t} = gdt$. The nominal rate i^* is assumed to equal a constant such that inflation is equal to zero, and the central bank holds no assets, $k_t^{CB} = 0$ for all t.

I conjecture a class of equilibria in which consumption C_t, output Y_t, the capital stock K_t, and funds' net worth N_{it} all grow at the same rate as productivity A_t. The price of capital q_t and labor supply ℓ_t are constant, and inflation is equal to zero. Moreover, along the BGP, flows into each fund are equal to zero, and funds follow their investment mandates ($\omega_{it} = \omega_i^*$ for all i and t). For variables that grow with productivity, lowercase letters denote quantities normalized by productivity A_t, i.e., $c_t \equiv \frac{C_t}{A_t}$, $y_t \equiv \frac{Y_t}{A_t}$, $k_t \equiv \frac{K_t}{A_t}$, and $n_{it} \equiv \frac{N_{it}}{A_t}$. Stars denote values of these variables along a BGP: $c_t = c^*$ for all t, $k_t = k^*$, and so on.

A capital stock that grows at a constant rate g implies $\phi(x^*) = \delta + g$, so (5) yields

$$q^* = \frac{1}{\nu} e^{\frac{\delta+g}{\nu}}. \tag{19}$$

Since consumption is cointegrated with productivity A_t along a BGP, its growth rate is g as well. The Euler equation (10) then implies

$$r^* = \rho + g, \tag{20}$$

where $r^* \equiv i^* - \pi^*$ denotes the (constant) real interest rate along the BGP. In a deterministic economy, the returns on capital (12) must be equal to the risk-free rate, so (19) yields

$$r^{k*} \equiv \frac{\alpha y^*}{q^* k^*} + g - \nu = r^*. \tag{21}$$

When capital and bonds earn the same returns, funds will have no incentive to deviate from their mandates. Likewise, households will have no incentive to reallocate their savings across funds (confirming the conjecture that flows are equal to zero).

Equating returns on capital and bonds, the steady-state level of capital must satisfy

$$k^* = \frac{\alpha}{\rho + \nu} \frac{y^*}{q^*}. \tag{22}$$

Then, using the resource constraint (2), it is possible to show that along a BGP, consumption and investment are constant fractions of output:

$$x^* k^* = \frac{\alpha \nu}{\rho + \nu} y^*, \; c^* = \frac{\rho + (1-\alpha)\nu}{\rho + \nu} y^*. \tag{23}$$

Finally, the BGP levels of labor and output,

$$\ell^* = \frac{(1-\alpha)(\rho + \nu)}{\rho + (1-\alpha)\nu} \quad \text{and} \quad y^* = k^{*\alpha} \ell^{*1-\alpha}, \tag{24}$$

follow from setting the marginal product of labor times the marginal utility of consumption equal to one (the marginal disutility of labor) and the production function in (2).

It remains to determine funds' net worth along the BGP. The market-clearing condition (13) implies that since the central bank holds no assets on the BGP, funds must hold the entire capital stock. Let $\theta_i^* \equiv \frac{n_i^*}{\sum_{i'=1}^{I} n_{i'}^*}$ denote the constant *fraction* of aggregate financial wealth held by fund i along the BGP. Then the market clearing condition implies

$$\sum_{i=1}^{I} \theta_i^* \omega_i^* = 1. \tag{25}$$

For each vector $\boldsymbol{\theta}^* = (\theta_1^*, \ldots, \theta_I^*)$ satisfying (25), there exists a BGP with constant wealth shares $\boldsymbol{\theta}^*$. Henceforth, I fix some such vector of wealth shares and log-linearize around a BGP with those wealth shares.

3.2 Log-linearizing around the BGP

Having derived the BGP equilibrium, I now take a log-linear approximation around it. Unless specified otherwise, a hatted variable will denote its (log-) deviation from its BGP level, e.g. $\hat{c}_t \equiv \frac{c_t - c^*}{c^*}, \hat{q}_t \equiv \frac{q_t - q^*}{q^*}$, $\hat{\omega}_{it} \equiv \frac{\omega_{it} - \omega_i^*}{\omega_i^*}$, etc. For rates of return, hats denote *level* deviations from steady state, e.g., $\hat{r}_t \equiv r_t - r^*, \hat{r}_t^k \equiv r_t^k - r^{k*}, \hat{i}_t \equiv i_t - i^*$. A dot denotes the drift of a variable, e.g., $\dot{\hat{c}}_t \equiv \frac{1}{dt} \mathbb{E}_t[d\hat{c}_t]$.

Away from the BGP, it is necessary to fully specify policy rules for the central bank. In order to study unanticipated changes in policy, I introduce sequences of exogenous nominal rate shocks $\{\nu_t^i\}$ and asset purchase shocks $\{\nu_t^k\}$. The (possibly stochastic) processes followed by these shocks are announced at $t = 0$. The nominal interest rate follows a Taylor rule with a constant nominal rate target i^* and a coefficient $\varphi > 1$ on inflation, so the deviation of the nominal rate from its target can be written as

$$\hat{i}_t = \varphi \hat{\pi}_t + \nu_t^i. \tag{26}$$

For analytical convenience, I take the limit $\varphi \to 1$. The Fisher equation can then be used to write the real rate simply as

$$\hat{r}_t = \nu_t^i. \tag{27}$$

I assume that the central bank does not make systematic asset purchases in the absence of shocks:

$$\hat{k}_t^{CB} = \nu_t^k. \tag{28}$$

In what follows, I first discuss the equilibrium in financial markets, which is the novel aspect of the model. Then, I outline equilibrium in goods markets, which is relatively standard.

Remark: The log-linear approximation takes the volatility σ of productivity shocks to be small, so that all terms of order $O(\sigma^2)$ are neglected. In particular, this implies that the risk premium on capital will be approximated as zero. Nevertheless, the model will be able to address *changes* in the premium on capital arising from funds' costs of deviating from their investment mandates. While this premium is not a risk premium in the usual sense, it is compensation that financial intermediaries require to hold risky assets rather than safe assets, so (with some abuse of terminology) I will sometimes refer to it as a risk premium regardless.

3.3 Equilibrium in asset markets

Let

$$\hat{\eta}_t \equiv \hat{r}_t^k - \hat{r}_t$$

denote the expected excess return on capital, which will be the key price that determines the *price-output ratio* $py_t \equiv \frac{q_t K_t}{Y_t}$ (i.e., the ratio of the value of the capital stock to output). The price-output ratio can be linearized as

$$\hat{py}_t \equiv \hat{q}_t + \hat{k}_t - \hat{y}_t.$$

This ratio will be an important determinant of investment demand, as demonstrated in the next section. This is the channel through which asset prices are connected to the real economy.

The asset pricing identity (12) can be linearized to obtain the Campbell-Shiller log-linearization of returns,

$$\hat{r}_t + \hat{\eta}_t = (\rho + \nu)(\hat{y}_t - \hat{q}_t - \hat{k}_t) + \dot{\hat{q}}_t + \dot{\hat{k}}_t.$$

Expected returns are high when either the dividend-price ratio is high (the first term on the right-hand side) or when expected capital gains are high (the second and third terms). As it turns out, this decomposition of returns can be combined with the Euler equation (10) and the resource constraint to obtain a pricing equation that determines the price-output ratio $\hat{py}_t$:

$$\frac{\rho + (1-\alpha)\nu}{\rho + \nu}\hat{\eta}_t = -(\rho + (1-\alpha)\nu)\hat{py}_t + \dot{\hat{py}}_t, \tag{29}$$

which can be solved forward as $\hat{py}_t = -\mathbb{E}_t\left[\int_0^\infty e^{-(\rho+(1-\alpha)\nu)s}\frac{\rho+(1-\alpha)\nu}{\rho+\nu}\hat{\eta}_{t+s}ds\right]$. That is, the price-output ratio tends to be low when the premium on capital is expected to be high in the future.

Hence, the excess return on capital will fully determine how conditions transmit from financial markets to the real economy. I show that in the short run, the excess return on capital in this model is determined by three factors: (1) the central banks' direct purchases of capital, (2) the aggregate elasticity of funds' demand for capital, and (3) fund' net worth shares

$$\theta_{it} \equiv \frac{n_{it}}{q_t k_t},$$

which are the key state variables in this economy. This result stands in stark contrast to typical representative agent models in which all assets are priced by the household's stochastic discount factor. Here, inelastic intermediary asset demand can cause the stochastic discount factor in financial markets to deviate from the household's, and central bank asset purchases can directly influence the returns on capital, at least in principle.

To first order, the optimal deviation of fund i's portfolio weight on capital from the benchmark ω_i^* can be written as

$$\hat{\omega}_{it} = \varepsilon_i \hat{\eta}_t \quad \text{where} \quad \varepsilon_i \equiv \frac{1}{\omega_i^* \xi_i''(0)}. \tag{30}$$

That is, fund i's portfolio weight on capital depends only on the excess return $\hat{\eta}_t$, and the elasticity ε_i of the fund's capital demand tends to be smaller when it faces a more convex cost ξ_i of deviating from its mandate. If the fund did not face a cost of deviating from its benchmark portfolio weight ($\varepsilon_i \to \infty$), the first-order condition would instead simply be $\hat{\eta}_t = 0$: the fund would be willing to hold any quantity of capital as long as it could earn a positive (expected) excess return. Inelastic asset demand at the fund level is therefore essential to generate fluctuations in the excess returns on capital.

The market clearing condition (13) can be log-linearized as $\sum_{i=1}^{I} \theta_i^* \omega_i^* (\hat{\theta}_{it} + \hat{\omega}_{it}) = -\hat{k}_t^{CB}$, where $\hat{k}_t^{CB} = \frac{k_t^{CB}}{k_t}$ denotes the fraction of the capital stock held by the central bank. Inserting funds' optimal portfolio weight (30), a key simplification becomes clear: the mixed funds $i \in \{1, \ldots, I\}$ aggregate to a *representative* mixed fund. Henceforth, I will refer to this representative fund as the "the market."

Proposition 3 (Aggregation)**.** *To first order, funds $i \in \{i, \ldots, I\}$ aggregate to a representa-*

tive fund with wealth share and demand elasticity

$$\hat{\theta}_t \equiv \sum_{i=1}^{I} (\theta_i^* \omega_i^*) \hat{\theta}_t, \ \varepsilon \equiv \sum_{i=1}^{I} (\theta_i^* \omega_i^*) \varepsilon_i.$$

The cost to the household of adjusting its holdings in the representative fund, and its target portfolio weight on capital, are given by

$$\chi \equiv \sum_{i=1}^{I} (\theta_i^* \omega_i^*) \chi_i, \ \omega^* \equiv \sum_{i=1}^{I} (\theta_i^* \omega_i^*) \omega_i^*.$$

The market's wealth share and demand elasticity are weighted averages of the corresponding objects for individual funds. The weight placed on each fund is equal to its *share of aggregate capital holdings*, so that larger funds and those that invest more aggressively in capital are more important in determining the aggregate market's demand for risky assets.

After aggregating, the market clearing condition can be written simply in terms of the (1) the central bank's capital purchases, (2) the market's wealth share, and (3) the market's demand elasticity.

$$\hat{\theta}_t + \varepsilon \hat{\eta}_t = -\hat{k}_t^{CB}. \tag{31}$$

The left-hand side of this equation can be read as a demand curve for capital. The first term $\hat{\theta}_t$ is a demand shifter: the demand for capital shifts outwards when the market holds a greater share of net worth. This is because the market's net worth dictates its *risk-bearing capacity*: since it aims to remain close to a target *portfolio weight* ω^* on capital, an increase in the market's capitalization raises its target capital holdings. The second term reflects the aggregate elasticity of capital demand with respect to the excess return $\hat{\eta}_t$: as the excess return on capital increases, the market will increase its capital demand. Hence, (31) shows that $\hat{\eta}_t$ is indeed determined by central bank asset purchases, the elasticity of the market's asset demand, and the market's wealth share, as claimed.

In the long run, however, households can undo the effects of central bank asset purchases by reallocating their savings across funds. For instance, if the central bank were to sell capital, increasing its excess return, households would find it attractive to shift their savings into the market. This logic is captured by the optimality condition for flows (11), which can be log-linearized and combined with the market's portfolio decision (30) to obtain

$$\rho \hat{f}_t = \frac{\omega^*}{\chi} \hat{\eta}_t + \dot{\hat{f}}_t, \tag{32}$$

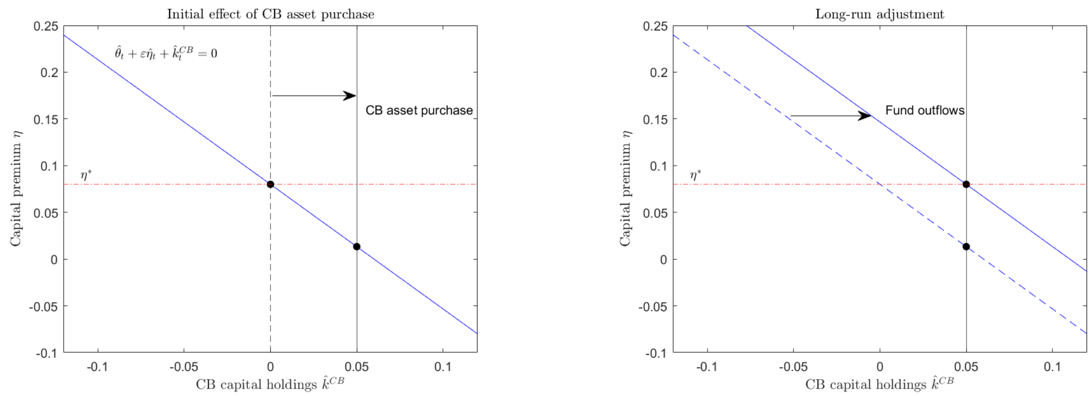

Figure 2: An illustration of the effects of a permanent asset purchase. Initially, the asset purchase reduces the supply of assets that the market has to absorb (i.e., the black line moves to the right). In the long run, households withdraw their funds from the market, shifting in the locus $\hat{\theta}_t + \varepsilon\hat{\eta}_t + \hat{k}_t^{CB} = 0$ (the blue line).

where $\hat{f}_t \equiv \frac{\sum_{i=1}^{I}(\theta_i^*\omega_i^*)F_{it}}{A_t}$ denotes flows into the market normalized by productivity A_t.

Note the similarity between the equation for optimal flows into the market and the linearized Phillips curve (37): just as firms raise prices when they anticipate the output gap to be positive in the future, the household shifts savings into the market when the excess return on the market's portfolio $\omega^*\hat{\eta}_t$ is expected to be high in the future. Moreover, just as interest rate policy is neutral in the long run in most Keynesian models, in this model, central bank asset purchases are neutral in the long run. Figure 2 illustrates why this is the case: following a permanent asset purchase that initially decreases the premium on capital, the household begins to gradually withdraw funds from the market until the returns on capital and bonds are equalized.

It remains to fully specify how the market's wealth share $\hat{\theta}_t$ evolves over time. The law of motion of $\hat{\theta}_t$ is given by

$$d\hat{\theta}_t = ((\omega^* - 1)\hat{\eta}_t - \hat{r}_t - (\rho + \nu)\hat{py}_t + \frac{1}{n^*}\hat{f}_t)dt + \sigma_{qt}dZ_t, \tag{33}$$

where σ_{qt} is the volatility of $\hat{q}_t$. The distribution of net worth evolves due to flows from the household, excess returns earned on capital, and stochastic return shocks: when the value of the capital stock appreciates, the market's net worth increases, since it holds a leveraged position in capital.

Equilibrium in asset markets is fully determined by (29)-(33), enabling an analytic characterization. A depressed level of the market's wealth share $\hat{\theta}_t$, i.e., the market's *risk-bearing* *capacity*, tends to increase the required returns on capital, thereby decreasing asset prices. In the absence of asset purchases, capital earns an excess premium when the market's risk-bearing capacity is depressed, $\hat{\eta}_t = -\frac{\hat{\theta}_t}{\varepsilon}$. The more inelastic the market's demand for risky assets, the greater the premium on capital. The excess premium on capital causes a decline in the price-output ratio. The elevated returns on capital, however, induce the household to reallocate its portfolio towards the market. Gradually, then, the market is recapitalized and the excess premium on capital is eliminated. The following proposition formalizes this result.

Proposition 4. *There exists* $\zeta > 0$ *such that starting from an initial state* $\hat{\theta}_0 < 0$ *(*$\hat{\theta}_0 > 0$*),*

- *The expected excess returns on capital at t positive (negative):* $\mathbb{E}_0[\hat{\eta}_t] = e^{-\zeta t}\frac{\hat{\theta}_0}{\varepsilon}$*;*
- *Expected flows into the market at t are positive (negative):* $\mathbb{E}_0[\hat{f}_t] = -e^{-\zeta t}\frac{1}{\rho+\zeta}\frac{\theta_0}{\chi\varepsilon}$*;*
- *The expected price-output ratio is depressed (elevated):* $\mathbb{E}_0[\hat{py}_t] = e^{-\zeta t}\frac{\theta_0}{\varepsilon(\rho+\nu(1-\alpha)+\zeta)}$.

Proposition 4 characterizes how changes in the market's risk-bearing capacity transmits to asset prices and flows as the economy transitions back to the BGP. I call this the *inelastic markets* channel of transmission, which will be a main focus throughout the rest of the analysis.

3.4 Equilibrium in goods markets

Next, I discuss how the inelastic markets channel shapes the equilibrium in goods markets. On the demand side, investment is determined by Tobin's Q, per (5). This equation can be log-linearized to obtain that total investment demand $x_t k_t = x^* k^*(1 + \hat{x}_t + \hat{k}_t)$ scales with the value of the capital stock:

$$\hat{x}_t + \hat{k}_t = \hat{q}_t + \hat{k}_t. \tag{34}$$

Using this expression for investment demand, it will be convenient to decompose the total demand for goods $\hat{y}_t = \frac{c^*}{y^*}\hat{c}_t + \frac{\delta k^*}{y^*}(\hat{x}_t + \hat{k}_t)$ in terms of consumption c_t and the price-output ratio $\hat{py}_t = \hat{q}_t + \hat{k}_t - \hat{y}_t$,

$$\hat{y}_t = \hat{c}_t + \frac{\alpha\delta}{\rho + (1-\alpha)\nu}\hat{py}_t. \tag{35}$$

The price-output ratio is determined by equilibrium in asset markets, as discussed in the previous section.

Consumption demand is determined exactly as in a standard New Keynesian model. The household's Euler equation (10) becomes simply:

$$\dot{\hat{c}}_t = \hat{r}_t, \tag{36}$$

The Euler equation for nominal bonds holds because the household pays no cost to adjust its bond holdings, so at the margin, it is indifferent between consumption in the present and saving in bonds to consume one period later. Just as in a standard New Keynesian model, consumption demand is driven by substitution effects: when the household expects bond returns to be high, it postpones consumption to the future.

The log-linearized Phillips curve derived from (6) is similar to that in a standard New Keynesian model. It implies that in equilibrium, inflation tends to be high when agents anticipate that in the future, output will exceed its natural level $k_t^{\alpha} \ell^{*1-\alpha}$:

$$\rho \hat{\pi}_t = \frac{\kappa}{1-\alpha}\left(\hat{c}_t + \frac{\alpha^2 \delta}{\rho + (1-\alpha)\nu} \hat{py}_t - \alpha \hat{k}_t\right) + \dot{\hat{\pi}}_t. \tag{37}$$

This equation can be integrated forward to yield

$$\hat{\pi}_t = \frac{\kappa}{1-\alpha} \mathbb{E}_t \left[\int_0^{\infty} e^{-\rho s} (\hat{c}_{t+s} + \frac{\alpha^2 \delta}{\rho + (1-\alpha)\nu} \hat{py}_{t+s} - \alpha \hat{k}_{t+s}) ds \right].$$

As usual, inflation is high when agents anticipate strong demand in the future (either consumption demand or investment demand).

A key feature of this model is the link between equilibrium in financial markets and equilibrium in goods markets. When the market's risk-bearing capacity $\hat{\theta}_t$ is depressed, the premium $\hat{\eta}_t$ on capital is high. The change in the premium on capital transmits to the real economy by increasing Tobin's Q and investment demand. The decrease in demand is deflationary, since output is below potential. Moreover, the decline in investment puts a persistent drag on the economy's productive capacity in the medium run. The following proposition summarizes how changes in the market's risk-bearing capacity transmits to macroeconomic outcomes through the inelastic markets channel.

Proposition 5. *If* $\hat{\theta}_0 < 0$, *then*

- *Expected output and investment are below their BGP levels for all* t, $\mathbb{E}_0[\hat{y}_t] < 0$ *and* $\mathbb{E}_0[\hat{x}_t + \hat{k}_t] < 0$;
- *The expected capital stock is below its BGP level for all* t, $\mathbb{E}_0[\hat{k}_t] < 0$;
- *Inflation is initially negative,* $\hat{\pi}_0 < 0$.

4 Monetary Policy Shocks

Having characterized the model's equilibrium and the novel inelastic markets channel, I now study unanticipated shocks to the monetary policy rule. The unanticipated shocks are assumed to decay exponentially at some rate λ,

$$\nu_t^i = e^{-\lambda t}\nu_0^i \quad \text{and} \quad \nu_t^k = e^{-\lambda t}\nu_0^k,$$

where ν_t^i is the shock to the Taylor rule (26) and ν_t^k is the shock to the asset purchase rule (28).

For both interest rate shocks and central bank asset purchase shocks, a useful decomposition precisely clarifies the role of *inelastic asset markets* in this model. The effects of any monetary policy shock can be decomposed into a *conventional channel* and an *inelastic markets channel*. The conventional channel is simply the shock's effect in a conventional New Keynesian model without asset market frictions (i.e., either households do not face portfolio adjustment frictions or funds do not pay a cost to deviate from their mandates). The inelastic markets channel captures the additional effect of the shock that can be attributed to financial markets' downwards-sloping asset demand, as illustrated in the previous section.

This channel of transmission depends only on the shocks to supply and demand in risky asset markets. Central bank asset purchases determine the *supply* of risky assets that financial markets must absorb. The *demand* for risky assets depends on the market's risk-bearing capacity (i.e., its wealth share). Both interest rate shocks and central bank asset purchases affect asset prices and therefore redistribute wealth. A change in asset prices at $t = 0$ from q^* to q_0 revalues the market's normalized net worth to $n_0 = \big(1 + \omega^*(\frac{q_0}{q^*} - 1)\big)n^*$, so the change in its wealth share at $t = 0$ is

$$\hat{\theta}_0 = (\omega^* - 1)\hat{q}_0. \tag{38}$$

An increase in capital prices ($\hat{q}_0 > 0$) *always* increases the market's risk-bearing capacity, since the market takes leverage to purchase capital ($\omega^* > 1$).[11]

The following proposition summarizes this decomposition of the channels through which monetary shocks affect the economy.

Proposition 6. *Given a sequence of monetary policy shocks* (ν_t^i, ν_t^k)*, the equilibrium response*

[11] Here the market must be levered because the bond fund holds a positive quantity of bonds and bonds are in zero net supply, so the market must borrow from the bond fund. Even if bonds were in positive net supply and the market were not levered, though, it would continue to be the case that an increase in capital prices would increase the market's wealth share.

of any variable $\hat{z}_t$ can be decomposed as

$$\hat{z}_t = \underbrace{\hat{z}_t^c}_{conventional} + \underbrace{\hat{z}_t^m}_{inelastic\ mkt},$$

where $\hat{z}_t^c$ is the response of $\hat{z}_t$ to the shock in a model without portfolio frictions and $\hat{z}_t^m$ depends only on the path of central bank asset purchases ν_t^k and the initial shock to the market's wealth share $\hat{\theta}_0$.

The remainder of the analysis in this section primarily studies how inelastic markets alter the transmission of monetary policy shocks to both financial markets and the real economy.

4.1 The transmission of interest rate shocks

I begin with the analysis of interest rate shocks in this economy ($\nu_0^i \neq 0$ but $\nu_0^k = 0$). The conventional channel of interest rate transmission is similar to that in any standard New Keynesian economy with capital: an interest rate cut ($\nu^i < 0$) stimulates the economy by boosting both consumption demand and investment. In turn, this stimulus leads to inflation. The accumulation of capital implies that the stimulative effects of the shock persist even after interest rates have returned to a normal level, since the economy's productive capacity remains elevated.

The inelastic markets channel *amplifies* the conventional channel of interest rate policy in the following sense.

Proposition 7. *An unanticipated interest rate cut $\nu_0^i < 0$ initially increases the market's risk-bearing capacity ($\hat{\theta}_0 > 0$). As a result, the expected premium on capital $\mathbb{E}_0[\hat{\eta}_t]$ is positive for all t. Furthermore,*

- *Expected output and investment $\mathbb{E}_0[\hat{y}_t], \mathbb{E}_0[\hat{x}_t + \hat{k}_t]$ are higher than in an economy without inelastic markets;*
- *Initial inflation $\hat{\pi}_0$ is higher than in an economy without inelastic markets.*

Figure 3 illustrates these dynamics, highlighting how the effect of an interest rate shock is decomposed into a conventional an an inelastic markets channel. An interest rate hike reduces the price of capital and redistributes wealth away from the market ($\hat{\theta}_t$ decreases). In turn, this reduction in the market's risk-bearing capacity increases the premium on capital, amplifying the initial fall in asset prices. Investment and output therefore both respond more strongly to an interest rate shock in this model than they would in a conventional New Keynesian economy without portfolio frictions. The initial boost to investment exacerbates

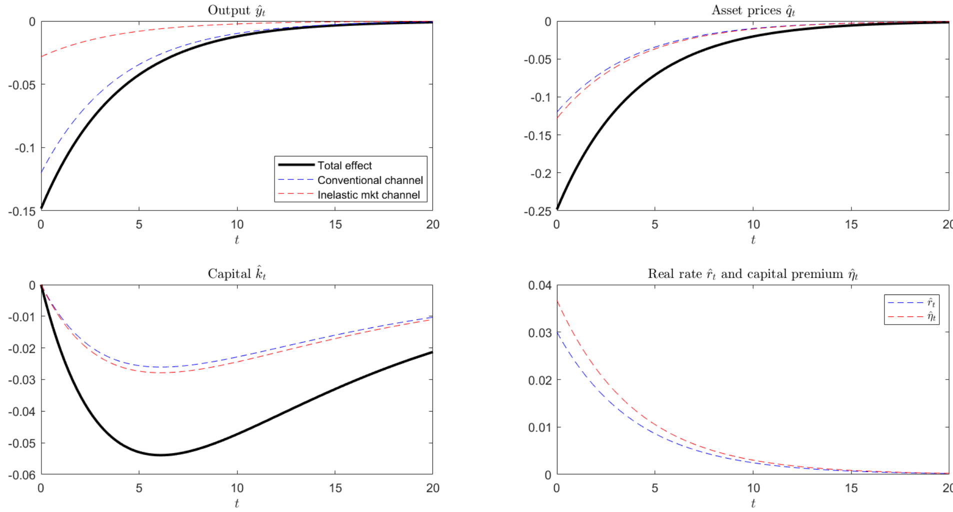

Figure 3: Dynamics of an unanticipated interest rate hike ($\nu_t^i = e^{-\lambda t}\nu^i$ with $\nu^i > 0$).

the decumulation of the capital stock, yielding a larger medium-run decrease in the economy's productive capacity.

Proposition 7 provides a rationalization for the well-known empirical fact that interest rate hikes (cuts) tend to coincide with increases (decreases) in the premia on risky assets, referred to as the "risk-taking channel" of monetary policy (Borio and Zhu, 2012). Observers have argued that interest rate hikes tend to reduce financial markets' "sentiment" or "risk appetite" (Kashyap and Stein, 2023); similarly, interest rate cuts have been said to induce "reach-for-yield" behavior in the financial sector that depresses risk premia (Bernanke, 2005).

Here, the fundamental reason why interest rate changes affect risk premia is that intermediaries have different propensities to take risk, leading some to be more exposed to interest rate shocks than others.[12]

4.2 Asset purchase shocks

Unlike in a conventional model, central bank asset purchases can have real effects when asset demand in financial markets is inelastic. In this section, I focus on a central bank asset purchase shock that is gradually unwound over time, $\nu^i = 0$ but $\nu^k > 0$.

[12]Similar explanations are provided by Brunnermeier and Sannikov (2016), who emphasize redistribution from intermediaries to households, and Kekre and Lenel (2022), who emphasize redistribution across *households* with different marginal propensities to take risk. However, this explanation is distinct from those based on transmission from liquidity premia to risk premia (e.g., Drechsler, Savov, and Schnabl 2018; Bigio and Sannikov 2021) or those based on distorted incentives to take risk in low-rate environments (e.g., Martinez-Miera and Repullo, 2017).

On impact, a central bank asset purchase causes a contraction in the supply of risky assets that the market must absorb, which reduces the premium on capital and raises asset prices (while increasing the market's wealth share). The increase in asset prices transmits to the real economy by boosting investment demand, which stimulates output and provides inflationary pressure.

Proposition 8. *A central bank asset purchase ($\nu_0^k > 0$) decreases the premium on capital on impact ($\hat{\eta}_0 < 0$) and increases the price-output ratio ($\hat{py}_0 > 0$). Output, investment, and inflation increase on impact as well, $\hat{y}_0 > 0, \hat{x}_0 + \hat{k}_0 > 0$, and $\hat{\pi}_0 > 0$.*

The long-run effect of an asset purchase is ambiguous, however. When the central bank purchases assets, the reduction in the premium on capital induces households to reallocate their savings away from the market and towards bonds. Even though the market's net worth increases at impact due to the appreciation of asset prices, the household's reallocation of savings drains the market's net worth and reduces its risk-bearing capacity in the long run. If the central bank's asset purchase depresses the premium on capital for long enough, the initial increase in the market's net worth is eventually reversed due to outflows. Then, once the central bank unwinds its balance sheet, the market's depressed risk-bearing capacity implies that it will require an *increase* in the premium on assets to absorb the excess supply. Asset prices can therefore *undershoot* their steady-state levels when the central bank maintains assets on its balance sheet for extended periods of time. Figure 4 illustrates these dynamics.

The effect of asset price undershooting is, of course, the opposite of the initial increase in asset prices. The decrease in investment demand triggers a recession and a deflation (which the central bank partially counteracts via a nominal rate cut). The following proposition summarizes these perverse long-run effects of a persistent central bank balance sheet expansion, captured by a small enough value of λ.

Proposition 9. *For a persistent enough asset purchase shock (small enough λ),*

- *The premium on capital is higher than its BGP level, and the price-output ratio is lower, for large enough t ($\mathbb{E}_0[\hat{\eta}_t] > 0$ and $\mathbb{E}_0[\hat{py}_t] < 0$);*
- *Output, investment, and inflation undershoot their BGP levels for large enough t ($\mathbb{E}_0[\hat{y}_t] < 0$, $\mathbb{E}_0[\hat{x}_t + \hat{k}_t] < 0$, and $\mathbb{E}_0[\hat{\pi}_0] < 0$.*

Proposition 9 sheds light on the ongoing debate about the effects of undoing the large-scale balance sheet expansions that central banks have undertaken in recent years. A central point of contention is whether balance sheet contraction will simply undo the effects of previous

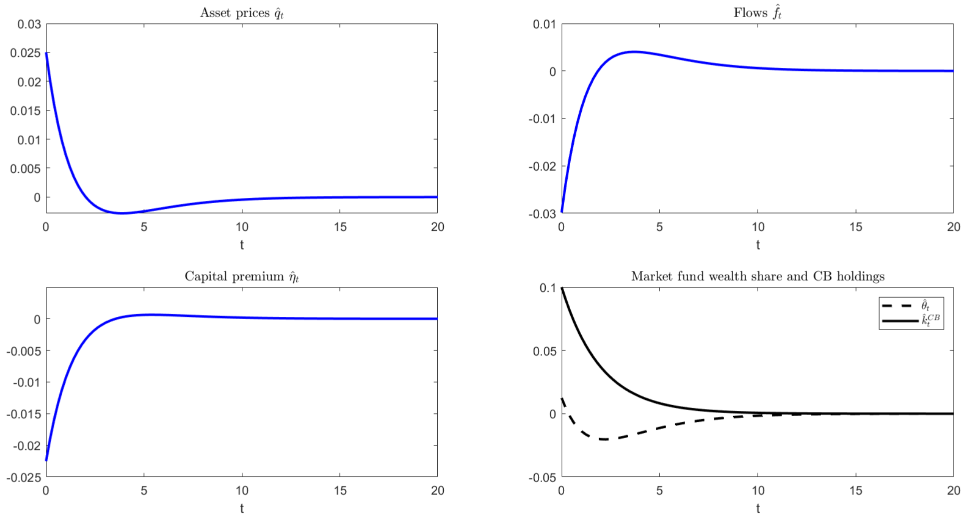

Figure 4: Dynamics of an unanticipated asset purchase ($\nu_t^k = e^{-\lambda t}\nu^k$ with $\nu^k > 0$).

stimulus in financial markets, or whether the prolonged period of large central bank balance sheets has led to the build-up of vulnerabilities in the financial sector that will then become apparent. For instance, Karadi and Nakov (2021) argue that a prolonged central bank balance sheet expansion can cause the financial sector to become undercapitalized, leading to a situation in which financial markets become "addicted to quantitative easing." In this model, there is a similar phenomenon: long balance sheet expansions depress returns in the financial sector, incentivizing households to withdraw their funds and enhancing financial fragility.

5 Optimal Policy

I now turn to the question of optimal policy. Whereas the previous sections addressed how the economy responds to *exogenous* changes in monetary policy, this section studies how the central bank should jointly manage interest rates and asset purchases in order to maximize welfare.

A *first-best* allocation in this economy is a set of processes $\{C_t, x_t, \ell_t, K_t\}$ that maximizes the household's utility $\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\Big(\log C_t - \ell_t\Big)dt\right]$ subject to the feasibility constraints $C_t + x_t K_t = A_t K_t^\alpha \ell_t^{1-\alpha}$ and $dK_t = (\phi(x_t) - \delta)K_t dt$. As it turns out, the optimality of an allocation is governed by two key quantities: an *output gap* and an *investment-output ratio*. In this model, the economy's natural level of output y_t^n (i.e., the level of output that would be produced without nominal rigidities or portfolio frictions) is proportional to K_t^α, so

$$y_t^n = k_t^\alpha \ell^{*1-\alpha}$$

where ℓ^* is defined in (24) and $k_t \equiv \frac{K_t}{A_t}$, as before. The optimal ratio of investment to output is

$$\frac{x_t k_t}{y_t} = \frac{\alpha\nu}{\rho+\nu}.$$

The following proposition summarizes this result.

Proposition 10. *An allocation* $\{C_t, x_t, \ell_t, K_t\}$ *is first-best if and only if* $y_t = k_t^\alpha \ell^{*1-\alpha}$ *and* $\frac{x_t k_t}{y_t} = \frac{\alpha\nu}{\rho+\nu}$ *for all* t.

In a competitive equilibrium, (5) implies that the investment-output ratio is given by $1+\nu p y_t$. Thus, the optimality of an equilibrium depends on the path of the output gap and the price-output ratio.

How can the central bank implement a first-best allocation?[13] As a benchmark result, I establish that (1) interest rate policy alone cannot implement the first-best, and (2) interest rate policy and asset purchases are jointly sufficient to implement the first-best.

Proposition 11. *In this economy,*

1. *(**No divine coincidence**): There does not exist a policy rule* $\{i_t, K_t^{CB}\}$ *with* $K_t^{CB} = 0$ *for all* t *that implements the first-best allocation.*

2. *(**Sufficiency of both instruments**): There exists a policy rule* $\{i_t, K_t^{CB}\}$ *that implements the first-best allocation.*

This proposition establishes that there is no "divine coincidence" in this economy. If the central bank sets the nominal rate to close the output gap at all times, it will not implement a first-best allocation. Exogenous shocks to the economy induce revaluations of assets that change the market's share of net worth, which in turn affects the premium on capital. Changes in asset values, in turn, affect the investment-output ratio, even if the central bank closes the output gap at all times. Specifically, there is (over-) underinvestment when the price-output ratio is (higher) lower than py^*.

The role of asset purchases is to neutralize these fluctuations in asset prices. In order to keep the price-output ratio at its optimal level, the central bank can purchase risky assets when the market's risk-bearing capacity is reduced and sell them when the market's risk

[13] Formally, policies $\{i_t, K_t^{CB}\}$ *implement* an allocation $\{C_t, x_t, \ell_t, K_t\}$ if that allocation arises in a competitive equilibrium with the given policies.

appetite is too high. Intuitively, while the nominal rate allows the central bank to address inefficiencies arising from nominal rigidities, asset purchases allow it to remedy the inefficiencies that arise due to portfolio rigidities. Hence, use of both policy tools suffices to implement the first-best.

To make the central bank's problem interesting, I assume that it is costly for the central bank to purchase (or short) risky assets. These costs could arise from inefficiencies in the central bank's management of loans to firms (as in Gertler and Karadi, 2011) or from political considerations that constrain the central bank's willingness to partially reduce the private sector's role in allocating capital. Formally, I assume that the central bank commits to a policy rule $\{i_t, K_t^{CB}\}$ to maximize

$$W = \mathbb{E}_0 \left[\int_0^\infty e^{-\rho s} \left(\log C_t - \ell_t - \lambda_k \big(\frac{K_t^{CB}}{K_t} \big)^2 \right) dt \right], \tag{39}$$

i.e., the cost incurred by the central bank is quadratic in the fraction of the capital stock $\frac{K_t^{CB}}{K_t}$ that it holds. In the following sections, I provide a tractable approximation to this problem and characterize the optimal policy.

5.1 Approximating the central bank's problem

To make the central bank's problem tractable, I log-linearize and take a second-order approximation to the welfare function. Moreover, for simplicity I take the limit as the household's portfolio adjustment costs go to infinity, $\chi_i \to \infty$ for all i. The key variables are the log output gap (denoted $\hat{\tilde{y}}_t$), the log price-output ratio (denoted $\hat{py}_t$), the log deviation of the market's share of net worth ($\hat{\theta}_t$), and the fraction of capital held by the central bank ($\hat{k}_t^{CB}$). As I show in the Appendix, the capital stock $\hat{k}_t$ actually drops out of the welfare function in the second-order approximation.[14] I take a Ramsey approach to the central bank's problem: I search for optimal processes for these variables subject to implementability constraints.

[14] This does not imply that changes in the capital stock do not affect welfare to second order. The welfare effects of fluctuations in the capital stock are captured in the parameters of the approximate policy problem.

The approximate policy problem can be written as

$$\max_{\hat{\tilde{y}}_t, \hat{py}_t, \hat{k}_t^{CB}, \hat{r}_t} \quad -\frac{1}{2}\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\left(\lambda_y \hat{\tilde{y}}_t^2 + \lambda_{py}\hat{py}_t^2 + \lambda_k(\hat{k}_t^{CB})^2\right)dt\right] \tag{40}$$

$$\text{s.t.} \quad \hat{\theta}_t + \hat{k}_t^{CB} + \frac{\rho + (1-\alpha)\nu}{\rho+\nu}\varepsilon\left(\dot{\hat{py}}_t - (\rho + \delta(1-\alpha))\hat{py}_t\right) = 0,$$

$$d\hat{\theta}_t = \left((\omega^* - 1)\frac{\rho + (1-\alpha)\nu}{\rho+\nu}(\dot{\hat{py}}_t - (\rho + (1-\alpha)\delta)\hat{py}_t) - (\rho+\nu)\hat{py}_t - \dot{\hat{\tilde{y}}}_t\right)dt$$

$$+ (\omega^* - 1)(\sigma_{py,t} + \sigma_{\tilde{y}t} + \sigma)dZ_t.$$

The central bank's loss function is quadratic (and separable) in the output gap, the price-output ratio, and asset purchases. The first constraint is the asset market clearing condition that relates the market's risk-bearing capacity $\hat{\theta}_t$ and central bank asset purchases $\hat{k}_t^{CB}$ to the risk premium, which (29) implies is dictated by the dynamics of the price-output ratio. The second constraint is simply a re-writing of the dynamics of the market's wealth share, which tends to increase when asset prices increase.

Separability implies that the optimal output gap and price-output ratio are independent from one another. A priori, it is therefore not obvious that the optimal interest rate policy (which controls the output gap) should depend in any way on financial fluctuations. In the next section, however, I demonstrate that in fact interest rate policy and asset purchases *do* optimally interact with one another, since both affect asset prices and therefore the market's risk-bearing capacity.

It is also important to note that this policy problem involves *forward-looking constraints*. The allocations that can be implemented at time t depend on agents' expectations of the future path of policy. In particular, the market's demand for assets and household portfolio flows hinge on expectations of the excess returns on capital, which in turn depend on future asset purchases. These forward-looking constraints appear in the policy problem via the presence of *expected* changes in the price-output ratio $\hat{py}_t$. Commitment is therefore an important feature of the central bank's problem, and allocations will be *history-dependent*: realized outcomes at time t will depend not only on the market's current risk-bearing capacity $\hat{\theta}_t$, but also on previous promises made by the central bank. From a technical standpoint, I use the recursive contracting tools developed by Marcet and Marimon (2019) to solve the central bank's problem.

5.2 Characterizing the optimal policy

This section characterizes the properties of the optimal policy and discusses the implications of the results. The key questions addressed by this analysis are (1) whether interest rate cuts and asset purchases should be used as complements or substitutes, and (2) how the central bank's asset purchase policy evolves over time in response to shocks.

The following proposition provides a simple characterization of the relationship between optimal interest rate policy and optimal asset purchases.

Proposition 12. *Under an optimal policy, the output gap is proportional to the quantity of assets purchased by the central bank:*

$$\hat{\hat{y}}_t = \frac{\lambda_k}{\lambda_y}(\omega^* - 1)\hat{k}_t^{CB}. \tag{41}$$

Proposition 12 demonstrates that interest rate cuts and asset purchases (or alternatively, interest rate hikes and asset sales) should be used as *complements*. The central bank cuts rates to provide additional stimulus (i.e., increases the output gap $\hat{\hat{y}}_t$) if and only if it purchases additional assets (increases $\hat{k}_t^{CB}$).

Why are the optimal output gap and quantity of asset purchases related? Indeed, why should the central bank *ever* permit output to deviate from the optimal level? Interest rate cuts and asset purchases are complements because both can be used to address frictions in asset markets. When the premium on capital is too high (so that the price-output ratio is too low), an interest rate cut boosts asset prices and therefore increases the market's risk-bearing capacity, reducing the premium. An asset purchase does double duty: it directly decreases the premium by reducing the quantity of assets the market has to absorb, and it also indirectly decreases the premium by raising asset prices and increasing the market's wealth share.

Note that as it becomes less costly for the central bank to hold assets ($\lambda_k \to 0$), the optimal size of the output gap for a given balance sheet size $\hat{k}_t^{CB}$ decreases. Intuitively, when the premium on capital is too high, the central bank's first resource is to expand its balance sheet in order to keep asset prices at their optimal level. However, at a certain point, the central bank finds it too costly to further expand its balance sheet. As a second-best option, the central bank can boost the market's risk-bearing capacity by cutting interest rates and allowing the economy to overheat. When it is costless to expand the balance sheet, though, there is no need to run the economy hot, so the central bank simply sets interest rates to close the output gap and simultaneously manages its balance sheet to target the price-output ratio.

It is worth noting that the key implication of Proposition 12 is that the *output gap* and asset purchases are proportional to one another. In this particular model, the "natural" level of interest rates does not depend on exogenous shocks, so this implies that an asset purchase is always accompanied by an interest rate cut. However, if the model were to include exogenous shocks that affect the natural rate, this would not necessarily be the case. For instance, consider an "impatience" shock that increases the household's discount rate ρ. The natural level of rates would rise, and the central bank would attempt to hike interest rates to ensure the economy does not overheat. The interest rate hike would cause a decrease in asset prices that would be partially remedied by an asset *purchase*, but the central bank would also refrain from hiking rates so much as to completely eliminate the output gap. In the face of shocks that change the natural rate, then, interest rate hikes and asset purchases can work in opposite directions.

Next, I characterize the optimal path of asset purchases. It is possible to show that the quantity of assets $\hat{k}_t^{CB}$ held by the central bank acts as a costate variable in the policy problem: it encodes the central bank's promises about the future path of asset prices.

Proposition 13. *Under the optimal policy, the path of asset purchases follows*

$$\dot{\hat{k}}_t^{CB} = -\zeta \hat{k}_t^{CB} - \lambda_p \hat{py}_t \tag{42}$$

for some $\zeta > 0$*.*

The central bank tends to sell assets when either the price-output ratio is above its optimal level ($\hat{py}_t > 0$) or when it already has a large balance sheet ($\hat{k}_t^{CB} > 0$). Note that even when the price-output ratio has returned to its optimal level, the central bank does not necessarily sell off its entire balance sheet. There are two reasons for this. Since asset demand is inelastic, an asset sale immediately causes a fall in asset prices, meaning that the price-output ratio would return to a suboptimally low level. Second, the central bank commits to smooth out the path of asset sales in order to guarantee *ex-ante* that asset prices will not fall too quickly in response to a favorable shock. Similarly, per Proposition 12, the central bank smooths the path of the output gap. This commitment (which is similar to a form of "forward guidance") is what boosts asset prices in the first place.

The market clearing equation (31) can be rearranged to obtain an asset pricing equation for the price-output ratio,

$$(\rho + (1-\alpha)\nu)\hat{py}_t = \frac{\hat{\theta}_t + \hat{k}_t^{CB}}{\varepsilon} + \dot{\hat{py}}_t, \tag{43}$$

which can be solved forward to obtain $\hat{py}_t = \mathbb{E}_t \left[\int_0^\infty e^{-(\rho+(1-\alpha)\nu)s} \frac{\theta_{t+s}+\hat{k}^{CB}_{t+s}}{\varepsilon} ds \right]$. That is, the price-output ratio is equal to discounted future asset demand $\hat{\theta}_t + \hat{k}^{CB}_t$ divided by the elasticity of asset demand ε. This asset pricing condition can be combined with (42) to precisely characterize the evolution of the price-output ratio.

Proposition 14. *Under the optimal policy, there exist positive constants p_θ, p_k such that*

$$\hat{py}_t = p_\theta \hat{\theta}_t + p_k \hat{k}^{CB}_t. \tag{44}$$

The price-output ratio is increasing in the market's risk-bearing capacity and in the central bank's asset purchases. A positive shock increases the market's wealth share $\hat{\theta}_t$ and causes an immediate jump in the price-output ratio. But then the optimal asset purchase rule (42) implies that the central bank gradually sells assets in order to correct the resulting inefficient overinvestment. At the same time, the central bank hikes interest rates to cool the economy (lowers $\hat{\tilde{y}}_t$), which counteracts the shock by limiting the increase in asset prices. Over time, the central bank gradually undoes its asset purchase and brings rates back to their natural level.

In summary, the optimal policy has two key qualities. First, interest rates are used in conjunction with asset purchases to correct inefficiencies in asset markets (at the cost of a non-zero output gap). Second, the central bank makes commitments to long-term asset purchases and only undoes its previous balance sheet operations slowly, even in the face of new shocks. The gradual, history-dependent nature of the central bank's optimal policy is precisely what permits that policy to be effective: asset prices are more sensitive to asset purchases when markets know those purchases will be long-lasting.

6 Conclusion

I develop a model of monetary policy with *inelastic markets*. The presence of intermediaries with downwards-sloping demand curves, as well as rigidities in household portfolio reallocation, imply that in the short run, asset prices are determined by financial markets' risk-bearing capacity and central bank asset purchases. Inelastic markets amplify the effect of conventional interest rate policy: by boosting asset prices, an interest rate cut raises intermediary net worth and therefore financial markets' risk-bearing capacity, which in turn raises the demand for risky assets and lowers risk premia. Central bank asset purchases can reduce risk premia and stimulate the economy in the short run, but if the central bank maintains a large balance sheet for too long, asset prices and output may undershoot once the asset purchase is unwound. In the long run, the household can reallocate its portfolio to undo the effects of central bank asset purchases, so asset purchases are neutral.

I apply the model to study the optimal joint conduct of interest rate policy and asset purchases. Interest rate policy alone cannot achieve the first-best: asset purchases are required to keep risk premia at their optimal level while interest rates change. The central bank's two tools are complements: when asset prices are too low, the central bank cuts rates to run the economy hot and purchases assets to reduce risk premia. The optimal path of asset purchases is also gradual: the central bank commits in advance to a prolonged asset purchase in order to increase the sensitivity of asset prices to the quantity purchased. Thus, a "late-exit" strategy from asset purchases during a recession is optimal.

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A Equilibrium

The equilibrium definition in this model is standard. The household, the firm, and funds optimize, and all markets clear. In this section, I derive the model's equilibrium equations, which I log-linearize in Section 3.

A.1 Supply side

The representative final goods producer solves the cost-minimization sub-problem

$$P_t^X = \min_{X_{jt}} \int_0^1 P_{jt} X_{jt} dj \ \text{ s.t. } \left(\int_0^1 X_{jt}^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} = 1.$$

As usual, the representative final goods producer's demand for good j is given by

$$\frac{X_{jt}}{X_t} = \left(\frac{P_{jt}}{P_t^X} \right)^{-\epsilon}.$$

The price index is

$$P_t^X = \left(\int_0^1 P_{jt}^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}}.$$

Then the representative final goods producer solves

$$\max_{K_t, X_t, x_t} P_t K_t^{\alpha} X_t^{1-\alpha} - P_t^X X_t - Ren_t^k K_t + \big(q_t \phi(x_t) - x_t\big) K_t.$$

The first-order conditions imply

$$P_t^X X_t = (1-\alpha) P_t Y_t, \ Ren_t^k K_t = \alpha P_t Y_t + P_t \big(q_t \phi(x_t) - x\big) K_t.$$

Moreover, as stated in the text,

$$q_t \phi'(x_t) = 1.$$

Monopolistic retailers $j \in [0,1]$ use intermediate goods as inputs to produce differentiated varieties (using the technology $X_{jt} = A_t \ell_{jt}$). Retailer j sets its price subject to Rotemberg-style quadratic adjustment costs: if it changes its nominal price P_{jt} by $dP_{jt} = \mu_{jt}^P P_{jt} dt$, it pays a cost $\frac{1}{2}\kappa \mu_{jt}^P$. The problem faced by producer j is then

$$\max_{\mu_{jt}^P} \mathbb{E}_0 \left[\int_0^{\infty} \left(\Lambda_t \left(\frac{P_{jt}}{P_t^X} \right)^{-\epsilon} \frac{P_{jt} - W_t/A_t}{P_t} Y_t - \frac{1}{2} \tilde{\kappa} {\mu_t^P}^2 \right) dt \right]$$

Let $p_{jt} \equiv \frac{P_{jt}}{P_t^X}$ denote the relative price of variety j to the composite intermediate good, let p_t^X denote the real price of the intermediate good, and let $mc_t = \frac{W_t/A_t}{P_t}$ denote the real marginal cost of production. The real price of variety j evolves according to

$$dp_{jt} = (\mu_{jt}^P - \pi_t^X) p_{jt} dt,$$

where $\pi_t^X = \frac{1}{dt}\mathbb{E}_t[\frac{dP_t^X}{P_t^X}]$ denotes expected inflation in the price of intermediate goods. The HJB equation associated with this problem is

$$\rho V(p_j, t) = \max_{\mu_j^P} \; \Lambda_t p_j^{-\epsilon}(p_j p_t^X - mc_t) Y_t - \frac{1}{2}\tilde{\kappa}\mu_j^{P2} + (\mu_j^P - \pi_t^X) p_j V_{p_j} + V_t.$$

The first-order condition is

$$\tilde{\kappa}\mu_{jt}^P = p_{jt} V_{p_j,t},$$

and the envelope theorem, coupled with the observation that $p_{jt} = 1$ for all t, implies

$$\rho V_{p_j,t} = \Lambda_t Y_t \big(\epsilon mc_t - (\epsilon - 1) p_t^X\big) + \frac{1}{dt}\mathbb{E}_t[dV_{p_z,t}],$$

which can be solved forward to obtain

$$V_{p_j,t} = \epsilon\, \mathbb{E}_t\left[\int_0^\infty e^{-\rho s}\Lambda_{t+s} Y_{t+s}\left(mc_{t+s} - \frac{\epsilon - 1}{\epsilon} p_{t+s}^X\right) ds\right].$$

Combining with the first-order condition, we have

$$\pi_t^X = \frac{\epsilon}{\tilde{\kappa}}\, \mathbb{E}_t\left[\int_0^\infty e^{-\rho s}\Lambda_{t+s} Y_{t+s}\left(mc_{t+s} - \frac{\epsilon - 1}{\epsilon} p_{t+s}^X\right) ds\right]. \tag{45}$$

A.2 The household's problem

The household's optimization problem can be written as an HJB equation with individual state variables (B_t, S_{it}). I denote the drift and volatility of transfers to the household by μ_{Tt}

and σ_{Tt}, respectively. The HJB equation is

$$\rho V(B,S,t) = \max_{C,\ell,F_i} \log C - \ell + \big(r_t B + \sum_{i=1}^{I} D_{it} S_i + \frac{W_t}{P_t}\ell + \mu_{Tt} - C - \sum_{i=1}^{I}(F_i + \frac{1}{2}\chi_{it}F_i^2)\big)V_B$$
$$+ \sum_{i=1}^{I}\big(F_i + (r_{it} - D_{it})S_i\big)V_{S_i} + \frac{1}{2}\sigma_{Tt}^2 V_{BB}$$
$$+ \frac{1}{2}\sum_{i,j}\sigma_{it}\sigma_{jt}S_iS_jV_{S_iS_j} + \sum_{i=1}^{I}\sigma_{Tt}\sigma_{it}S_iV_{BS_i} + V_t$$

The first-order condition with respect to C, $V_B = \frac{1}{C}$, yields a standard Euler equation. The Envelope Theorem implies

$$\rho V_B = r_t V_B + \mathbb{E}_t[dV_B],$$

with $dV_B = dC^{-1}$. We have

$$dC^{-1} = -\frac{1}{C}\left(\frac{dC}{C} - \frac{dC^2}{C^2}\right) = -\frac{1}{C}\big((\mu_C - \frac{1}{2}\sigma_C^2)dt + \sigma_C dZ\big),$$

so

$$r_t = \rho + \mu_{Ct} - \sigma_{Ct}^2,$$

as usual.

The first-order condition for flows F_i can be written as

$$V_{S_i} = (1 + \chi_{it}F_i)V_B. \tag{46}$$

Let $\xi_{it} \equiv V_{S_it}(B_t, S_t, t)$, $\lambda_t \equiv V_B(B_t, S_t, t)$. The first step is to show that

$$\rho\xi_{it} = \lambda_t D_{it} + \mu_{\xi_it} + (r_{it} - D_{it})\xi_{it} + \sigma_{\xi_it}\sigma_{it}. \tag{47}$$

Note that

$$\sigma_{\xi_it} = \sigma_{Tt}\frac{\partial \xi_{it}}{\partial B} + \sum_{j=1}^{I}\sigma_{jt}S_j\frac{\partial \xi_{it}}{\partial S_j}.$$

Similarly,

$$\sigma_{\lambda t} = \sigma_{Bt}\frac{\partial \lambda_t}{\partial B} + \sum_{i=1}^{I}\sigma_{it}S_{it}\frac{\partial \lambda_t}{\partial S_i}.$$

The Envelope Theorem applied to S_i implies

$$\rho V_{S_i} = D_{it} V_B + (r_{it} - D_{it}) V_{S_i} + \sigma_{Tt}\sigma_{it} V_{BS_i} + \frac{1}{2}\sigma_{it}^2 S_i V_{S_i S_i} + \sum_{j \neq i} \sigma_{jt}\sigma_{it} S_j V_{S_j S_i} + \frac{1}{dt}\mathbb{E}_t[dV_{S_i}]$$

By inspection, this is the same as (47).

Next, we can substitute (46), which can be rewritten as $\xi_{it} = (1 + \chi_{it} F_{it})\lambda_t$, in (47) to obtain

$$\rho\lambda_t \chi_{it} F_{it} = \lambda_t D_{it} - \rho\lambda_t + \frac{1}{dt}\mathbb{E}_t[\frac{d(\lambda_t X_{it})}{X_{it}}] + \frac{1}{dt}\mathbb{E}_t[\frac{d(\lambda_t \chi_{it} F_{it} X_{it})}{X_{it}}],$$

where

$$\frac{dX_{it}}{X_{it}} \equiv dR_{it} - D_{it}dt = (r_{it} - D_{it})dt + \sigma_{it} dZ_t.$$

This is because $\mu_{\xi_i t} + (r_{it} - D_{it})\xi_{it} + \sigma_{\xi_i t}\sigma_{it} = \mathbb{E}_t[\frac{d(\xi_{it} X_{it})}{X_{it}}]$.

We have

$$-\rho\lambda_t + \mathbb{E}_t[\frac{d\lambda_t X_{it}}{X_{it}}] = -r_t\lambda_t + (r_{it} - D_{it})\lambda_t + \sigma_{\lambda t}\sigma_{it},$$

using the Euler equation $\rho\lambda_t = r_t\lambda_t + \frac{1}{dt}\mathbb{E}_t[d\lambda_t]$. Then, letting

$$v_{it} \equiv \lambda_t \chi_{it} F_{it}$$

, we have

$$\rho v_{it} = \lambda_t r_{it} + \sigma_{\lambda t}\sigma_{it} - \lambda_t r_t + \mathbb{E}_t[\frac{d(v_{it} X_{it})}{X_{it}}]. \tag{48}$$

From this equation, it immediately follows that

$$v_{it} = \lambda_t \chi_{it} F_{it} = \mathbb{E}_t\left[\int_0^\infty e^{-\rho s}\frac{X_{i,t+s}}{X_{it}}\lambda_{t+s}(r_{i,t+s} - \sigma_{C,t+s}\sigma_{i,t+s} - r_{t+s})ds\right], \tag{49}$$

since $\sigma_{\lambda t} = -\lambda_t \sigma_{Ct}$.

A.3 Funds' problem

The marginal utility Λ_t that appears in funds' problem (16) evolves according to

$$\frac{d\Lambda_t}{\Lambda_t} = -(\mu_{Ct} - \sigma_{Ct}^2) - \sigma_{Ct} dZ_t,$$

where μ_{Ct}, σ_{Ct} denote the drift and volatility of consumption, respectively. The dividends paid by capital at t are denoted D_{it}. I conjecture and verify that the fund's value function at time t is affine, $V_{it}(N) = v_{it}N + z_t$. The fund's problem can then be written as an HJB equation:

$$(\rho+\mu_{Ct}-\sigma_{Ct}^2-\sigma_{Ct}\sigma_{v_it}-\mu_{v_it})v_{it}=\max_{\omega_i} r_t+\Big(\omega_i(r_t^k-r_t)-\xi(\omega_i-\omega_i^*)+\omega_i\sigma_{qt}(\sigma_{v_it}-\sigma_{Ct})\Big)v_{it},$$

where μ_{v_it} denotes the drift of v_{it} and σ_{v_it} denotes its volatility.

The first-order condition is

$$\xi'(\omega_{it}-\omega_i^*)=r_t^k-r_t-\sigma_{qt}(\sigma_{Ct}-\sigma_{vt}). \tag{50}$$

This first-order condition can be inverted to determine ω_t.

With the payout rate $\gamma_t=r_t$, the fund's net worth evolves according to

$$dN_t=(\omega_{it}(r_t^k-r_t)N_t+F_t)dt+\omega_{it}\sigma_{qt}N_tdZ_t.$$

The evolution of the fund's wealth share $\theta_{it}=\frac{N_{it}}{q_tK_t}$ then follows

$$\frac{d\theta_{it}}{\theta_{it}}=\frac{dN_{it}}{N_{it}}-\frac{d(q_tK_t)}{q_tK_t}+\frac{d(q_tK_t)^2}{(q_tK_t)^2}-\frac{dN_{it}}{N_{it}}\frac{d(q_tK_t)}{q_tK_t}$$

by Ito's lemma. This yields

$$\frac{d\theta_{it}}{\theta_{it}}=\left((\omega_{it}-1)(r_t^k-r_t-\sigma_{qt}^2)+\frac{F_{it}}{N_{it}}-r_t+\nu+\alpha\frac{Y_t}{V_t}\right)dt+(\omega_{it}-1)\sigma_{qt}dZ_t. \tag{51}$$

Here, I use the expression for the rental rate of capital (4) and the optimality condition for investment (5) to replace the drift of q_tK_t by $r_t^kdt-\frac{Ren_t^kK_t}{q_tK_t}dt$.

B Log-linear equilibrium

B.1 Deriving the BGP

In this section, I detail the economy's evolution along the deterministic balanced growth path with $\sigma=0$.

First, start with the household's Euler equation. We have

$$r^*=\rho+g. \tag{52}$$

Moreover, the return on capital is equal to the risk-free rate:

$$r^{k*}=r^*=\rho+g. \tag{53}$$

Investment must satisfy

$$\phi(x^*) = \delta + g \quad \Rightarrow q^* = \frac{1}{\nu} e^{\frac{\delta+g}{\nu}}, \tag{54}$$

using (5). Additionally, we have

$$y^* = c^* + \nu q^* k^*, \tag{55}$$

using the fact that $\phi(x^*)k^* = \nu q^* k^*$.

Along the balanced growth path, the firm's total net cash flow is $\alpha y^* - x^* k^* = \alpha y^* - \nu q^* k^*$, and capital gains accrue at rate g, so

$$r^{k*} = \alpha y^* q^* k^* - \nu + g.$$

Combining this expression with (53) and (55), we obtain

$$c^* = \frac{\rho + (1-\alpha)\nu}{\rho + \nu} y^*, \quad \nu q^* k^* = \frac{\alpha \nu}{\rho + \nu} y^*. \tag{56}$$

The Keynesian subsidy τ^ℓ is set so that labor supply is efficient along the BGP. This implies that wages times the marginal utility of consumption is equal to one, the marginal disutility of labor:

$$\frac{w^*}{c^*} = 1.$$

Hence, (4) implies

$$w^* \ell^* = (1-\alpha) y^*,$$

so

$$\ell^* = \frac{(1-\alpha)(\rho+\nu)}{\rho + (1-\alpha)\nu}, \tag{57}$$

using (56). We also have that the dividend yield on capital must be $\rho + \nu$, so

$$k^* = \left(\frac{\alpha}{(\rho+\nu) q^*} \right)^{\frac{1}{1-\alpha}} \ell^* \tag{58}$$

B.2 Log-linearization of main equilibrium conditions

This section provides a log-linear approximation to the equilibrium conditions. Lowercase variables denote variables normalized by productivity A_t, and hatted variables denote (log-)deviations from balanced growth path values.

Investment: The capital accumulation equation

$$dk_t = (\phi(x_t) - \delta - g)k_t dt = (\nu \log(\nu q_t) - \delta - g)k_t$$

linearizes to

$$\dot{\hat{k}}_t = \nu \hat{q}_t. \tag{59}$$

The household's problem: The household's Euler equation is simply

$$\hat{r}_t = \dot{\hat{c}}_t. \tag{60}$$

where $\hat{r}_t \equiv r_t - r^*$. The optimality condition for flows into fund i linearizes to

$$\rho \hat{f}_{it} = \frac{\omega_i^*}{\chi_i}(\hat{r}_t^k - \hat{r}_t) + \dot{\hat{f}}_{it}. \tag{61}$$

where $\hat{f}_{it} \equiv f_{it}$. This follows from the fact that along the BGP, $\frac{C_t}{C_{t+s}} = e^{-gs}$, $\frac{X_{i,t+s}}{X_{it}} = e^{gs}$, and $r_t^k = r_t$ (so that all log-deviations other than $\hat{r}_t^k - \hat{r}_t$ drop out of the linearized optimality condition).

Financial markets: The expected return on capital $\hat{r}_t^k$ is linearized in the main text. To obtain the pricing equation for the price-output ratio $\hat{py}_t$, observe that the resource constraint implies

$$\begin{aligned}\hat{y}_t &= \frac{c^*}{y^*}\hat{c}_t + \frac{\nu k^*}{y^*}(\hat{q}_t + \hat{k}_t) \\ &= \frac{c^*}{y^*}\hat{c}_t + \frac{\nu k^*}{y^*}(\hat{y}_t + \hat{py}_t) \\ &= \hat{c}_t + \frac{\nu q^* k^*}{c^*}\hat{py}_t,\end{aligned}$$

where $\frac{\nu k^*}{c^*} = \frac{\alpha\nu}{\rho + (1-\alpha)\nu}$. Then the log-linear Campbell-Shiller decomposition implies

$$\hat{\eta}_t = -(\rho + \nu)\hat{py}_t + \dot{\hat{q}}_t + \dot{\hat{k}}_t - \dot{\hat{c}}_t$$

after subtracting $\hat{r}_t$ from both sides and using the Euler equation. Then, replacing $\hat{q}_t + \hat{k}_t - \hat{c}_t = \frac{\rho+\nu}{\rho+(1-\alpha)\nu}\hat{py}_t$,

$$\frac{\rho + (1-\alpha)\nu}{\rho + \nu}\hat{\eta}_t = -(\rho - (1-\alpha)\nu)\hat{py}_t + \dot{\hat{py}}_t. \tag{62}$$

The market clearing condition (13) can be written as

$$\sum_{i=1}^{I} \omega_{it}\theta_{it} + \frac{k_t^{CB}}{k_t} = 1,$$

which linearizes to

$$\sum_{i=1}^{I} \theta_i^* \omega_i^* (\hat{\theta}_{it} + \hat{\omega}_{it}) + \hat{k}_t^{CB} = 0. \tag{63}$$

Funds: The first-order condition for the fund i's portfolio choice (50) becomes

$$\hat{\omega}_{it} = \frac{1}{\omega_i^* \xi_i''(0)} (\hat{r}_t^k - \hat{r}_t), \tag{64}$$

so we can define

$$\varepsilon_i \equiv \frac{1}{\omega_i^* \xi_i''(0)}$$

to be the fund's elasticity of demand for risky assets.

The fund's wealth share evolves according to (51) which log-linearizes to

$$d\hat{\theta}_{it} = (\omega_i^* - 1)(\hat{\eta}_t dt + \sigma_{qt} dZ_t) + \big(\frac{\hat{f}_{it}}{n_i^*} - \hat{r}_t - (\rho + \nu)\hat{py}_t\big) dt, \tag{65}$$

as in the main text.

Now it is possible to prove the fund aggregation result.

Proof of Proposition 3. Define flows $\hat{f}_t$ into the representative fund as

$$\hat{f}_t = \sum_{i=1}^{I} \theta_i^* \omega_i^* \hat{f}_{it}.$$

Aggregating the laws of motion (65), the law of motion of $\hat{\theta}_t$ can be written as

$$d\hat{\theta}_t = (\omega^* - 1)(\hat{\eta}_t dt + \sigma_{qt} dZ_t) + \big(\frac{\hat{f}_t}{n^*} - \hat{r}_t - (\rho + \nu)\hat{py}_t\big) dt,$$

where $\frac{1}{n^*} = \sum_{i=1}^{I} \theta_i^* \omega_i^* \frac{1}{n_i^*}$. The evolution of flows $\hat{f}_t$ satisfies

$$\rho \hat{f}_t = \frac{\omega^*}{\chi^*} \hat{\eta}_t + \dot{\hat{f}}_t,$$

where

$$\frac{\omega^*}{\chi^*} \equiv \sum_{i=1}^{I} \theta_i^* \omega_i^* \frac{\omega_i^*}{\chi_i}.$$

Moreover, the market-clearing condition (13) can be written as

$$\hat{\theta}_t + \varepsilon \hat{\eta}_t = -\hat{k}_t^{CB},$$

as desired. □

B.3 The transition path

The two state variables in this economy are the market's wealth share $\hat{\theta}_t$ and the capital stock $\hat{k}_t$. This section derives the economy's transition path from an arbitrary initial state $(\hat{\theta}_0, \hat{k}_0)$ back to the BGP.

Proof of Proposition 4. I guess and verify that the variables $(\hat{\theta}_t, \hat{py}_t, \hat{\eta}_t, \hat{f}_t)$ decay exponentially at rate ζ from their initial values (e.g., $\hat{\theta}_t = e^{-\zeta t}\hat{\theta}$). Under this conjecture, Equations (29)-(33) become

$$\begin{aligned} \text{(Asset pricing)}: &\ \hat{\eta} = -(\rho + (1-\alpha)\nu + \zeta)\hat{py}; \\ \text{(Market clearing)}: &\ \hat{\theta} + \varepsilon\hat{\eta} = 0; \\ \text{(Flows)}: &\ (\rho + \zeta)\hat{f} = \frac{\omega^*}{\chi}\hat{\eta}; \\ \text{(Law of motion)}: &\ -\zeta\hat{\theta} = (\omega^* - 1)\eta + \frac{1}{n^*}\hat{f} - (\rho + \nu)\hat{py}. \end{aligned}$$

These equations can be combined to show that the decay rate ζ is the solution to

$$\zeta\varepsilon(\rho + (1-\alpha)\nu + \zeta) = (\omega^* - 1)(\rho + (1-\alpha)\nu + \zeta) + \rho + \nu + \frac{\omega^*}{n^*\chi}\frac{\rho + (1-\alpha)\nu + \zeta}{\rho + \zeta}.$$

By inspection, $\hat{py}$ has the same sign as $\hat{\theta}$, which has the opposite sign as $\hat{\eta}$. Flows $\hat{f}$ have the same sign as $\hat{\eta}$, proving the result. □

Proof of Proposition 5. Note that since the real rate is constant, the Euler equation implies $\hat{c}_t = 0$ for all t. I guess and verify that the capital stock and inflation obey

$$\hat{k}_t = e^{-\zeta t}\hat{k}^{\zeta} + e^{-\nu t}\hat{k}^{\nu},$$

$$\hat{\pi}_t = e^{-\zeta t}\hat{\pi}^{\zeta} + e^{-\delta t}\hat{\pi}^{\delta}.$$

Equation (59), which describes the evolution of the capital stock, implies

$$\dot{\hat{k}}_t = \nu(\hat{py}_t + \hat{y}_t - \hat{k}_t).$$

The resource constraint and $\hat{c}_t = 0$ then imply

$$\dot{\hat{k}}_t = \nu(\frac{\rho+\nu}{\rho+(1-\alpha)\nu}\hat{py}_t - \hat{k}_t).$$

Plugging in the guess for $\hat{k}_t$,

$$\hat{k}^\zeta = \frac{\nu}{\nu-\zeta}\frac{\rho+\nu}{\rho+(1-\alpha)\nu}\hat{py}.$$

The initial condition for capital implies $\hat{k}^\delta = \hat{k}_0 - \hat{k}^\zeta$.

Investment and output always have the same sign as $\hat{py}_t$. To see that the capital stock has the same sign as $\hat{py}_0$, we need to consider two cases. First, suppose that $\zeta > \nu$. Then $\hat{k}^\zeta < 0$ and $\hat{k}^\nu = -\hat{k}^\zeta > 0$. But then

$$\begin{aligned}\hat{k}_t &= e^{-\zeta t}\hat{k}^\zeta + e^{-\nu t}\hat{k}^\nu \\ &> e^{-\nu t}\hat{k}^\zeta + e^{-\nu t}\hat{k}^\nu = 0,\end{aligned}$$

where the inequality follows from the fact that $\zeta > \nu$ and $\hat{k}^\zeta < 0$. The case $\zeta < \nu$ is analogous.

Inflation obeys

$$\rho\hat{\pi}_t = \frac{\kappa\alpha}{1-\alpha}(\frac{\alpha\delta}{\rho+(1-\alpha)\delta}\hat{py}_t - \hat{k}_t) + \dot{\hat{\pi}}_t.$$

Using the conjecture for $\hat{\pi}_t$,

$$(\rho+\zeta)e^{-\zeta t}\hat{\pi}^\zeta + (\rho+\delta)e^{-\delta t}\hat{\pi}^\delta = \frac{\kappa\alpha}{1-\alpha}(\frac{\alpha\delta}{\rho+(1-\alpha)\delta}e^{-\zeta t}\hat{py} - e^{-\zeta t}\hat{k}^\zeta - e^{-\delta t}\hat{k}^\delta).$$

Then

$$\hat{\pi}^\delta = -\frac{1}{\rho+\delta}\frac{\kappa\alpha}{1-\alpha}\hat{k}^\delta,$$

and

$$\hat{\pi}^\zeta = \frac{1}{\rho+\zeta}\frac{\kappa\alpha}{1-\alpha}\left(\frac{\alpha\delta}{\rho+(1-\alpha)\delta}\hat{py} - \hat{k}^\zeta\right).$$

If $\hat{k}_0 = 0$, then $\hat{\pi}_0$ is proportional to $\hat{py}_0$, as desired. □

C The economy's response to monetary shocks

In this section, I analytically characterize the economy's response to unanticipated monetary shocks $(\nu_t^i, \nu_t^k) = (e^{-\lambda t}\nu^i, e^{-\gamma t}\nu^k)$. I use a guess-and-verify method to find a solution to the equilibrium equations. Guess a solution of the form

$$\hat{z}_t = \hat{z}_t^P + \hat{z}_t^H,$$

where $\hat{z}_t^P$ is a particular solution (with $\nu_t^i, \nu_t^k \neq 0$) and $\hat{z}_t^H$ is a homogeneous solution (with $\nu_t^i, \nu_t^k = 0$). The particular solution is of the form

$$\hat{z}_t^P = e^{-\lambda t}\hat{z}^P,$$

whereas the homogeneous solution takes the form

$$\hat{z}_t^H = e^{-\beta_c t}\hat{z}_c + e^{-\beta_f t}\hat{z}_f.$$

Given a particular solution, there exists a unique homogeneous solution such that $\hat{k}_0^H + \hat{k}_0^P = 0$ and $\hat{\theta}_0^H + \hat{\theta}_0^P = (\omega^* - 1)(\hat{q}_0^P + \hat{q}_0^c + \hat{q}_0^f)$.

The homogeneous solution: I show that the homogeneous solution can be summarized by two eigenvectors $(\hat{k}^c, \hat{\theta}^c)$ and $(\hat{k}^f, \hat{\theta}^f)$. For the conventional solution, additionally, $\hat{\theta}^c = 0$. The conventional solution then satisfies

$$\hat{c}^c = \hat{y}^c = \hat{py}^c = 0,$$

$$-\beta_c \hat{k}^c = -\nu \hat{k}^c,$$

$$(\rho + \beta^c)\hat{\pi}^c = -\frac{\kappa\alpha}{1-\alpha}\hat{k}^c,$$

so $\beta^c = \nu$.

The other eigenvector $(\hat{k}^f, \hat{\eta}^f)$ was solved for in the previous section. The corresponding eigenvalue is $\beta_f = \zeta$.

C.1 Interest rate shocks

Consider an interest rate shock $\nu_t^i = e^{-\lambda t}\nu^i$. There are three solutions to consider: the particular solution, the homogeneous solution corresponding to the conventional channel, and the particular solution corresponding to the inelastic markets channel.

Particular solution for interest rate shocks: A particular solution satisfies

$$-\lambda\hat{c}^p = \nu^i,$$

$$-\lambda\hat{k}^p = \nu(\hat{c}^p - \hat{k}^p),$$

$$(\rho+\lambda)\hat{\pi}^p = \frac{\kappa}{1-\alpha}(\hat{c}^p - \alpha\hat{k}^p).$$

Here we have guessed that $\hat{\theta} = 0$. Indeed, it will be the case that $\hat{\theta} = 0$ for *all* particular solutions. Note that under this solution, output satisfies

$$\hat{y}^p = \hat{c}^p$$

by (35).

Under the full solution, we must have

$$\hat{\theta}^p + \hat{\theta}^c + \hat{\theta}^f = (\omega^* - 1)(\hat{y}^p + \hat{y}^c + \hat{y}^f + \hat{py}^p + \hat{py}^c + \hat{py}^f).$$

All objects on the right-hand side have been solved for above: $\hat{y}^p = -\frac{\nu^i}{\lambda}$, $\hat{y}^c = \hat{py}^c = \hat{py}^p = 0$, and $\hat{y}^f = \frac{\rho+\nu}{\rho+(1-\alpha)\nu}\hat{py}^f = \frac{\rho+\nu}{\rho+(1-\alpha)\nu}\frac{\hat{\theta}^f}{\varepsilon(\rho+(1-\alpha)\nu+\zeta)}$. Then

$$\left(1 - \frac{\rho+\nu}{\rho+(1-\alpha)\nu}\frac{\omega^*-1}{\varepsilon(\rho+(1-\alpha)\nu+\zeta)}\right)\hat{\theta}^f = -(\omega^*-1)\frac{\nu^i}{\lambda},$$

whereas $\hat{\theta}^p = \hat{\theta}^c = 0$. The coefficient on the right-hand side must be positive for an equilibrium to exist. Therefore, $\hat{\theta}^f$ and ν^i have opposite signs, which implies that if $\nu^i < 0$, then $\hat{\theta}^f > 0$ and $\hat{y}_0 = \hat{y}^p + \hat{y}^f > \hat{y}^p > 0$, $\hat{py}_0 = \hat{py}^f > 0$.

Proof of Proposition 7. The proposition is proven by the calculations above. □

C.2 Asset purchase shocks

Particular solution for asset purchase shocks: Now consider an asset purchase shock, $\nu^k > 0$ but $\nu^i = 0$. The particular solution does not satisfy $\hat{\theta} = 0$, since the risk premium will be different from zero. That is, under the particular solution, there are non-trivial flows. Due to this property, asset prices can undershoot when the asset purchase is undone.

The particular solution satisfies

$$\begin{aligned}
\text{(Asset pricing)}:&\ \ \hat{\eta} = -(\rho + (1-\alpha)\nu + \lambda)\hat{py};\\
\text{(Market clearing)}:&\ \ \hat{\theta} + \varepsilon\hat{\eta} = -\nu^k;\\
\text{(Flows)}:&\ \ (\rho+\lambda)\hat{f} = \frac{\omega^*}{\chi}\hat{\eta};\\
\text{(Law of motion)}:&\ \ -\lambda\hat{\theta} = (\omega^*-1)\hat{\eta} + \frac{1}{n^*}\hat{f} - (\rho+\nu)\hat{py};\\
\text{(k accum.)}:&\ \ (\nu-\lambda)\hat{k} = \nu\frac{\alpha\nu}{\rho+(1-\alpha)\nu}\hat{py}.
\end{aligned}$$

These equations can be simplified to obtain

$$\left(\frac{\omega^*-1}{\varepsilon}+\frac{1}{\rho+\lambda}\frac{\omega^*}{n^*\chi\varepsilon}+\frac{\rho+\nu}{\varepsilon(\rho+(1-\alpha)\nu+\lambda)}\right)\hat{k}^{CB} = \left(\lambda-\frac{\omega^*-1}{\varepsilon}-\frac{1}{\rho+\lambda}\frac{\omega^*}{\varepsilon\chi n^*}-\frac{\rho+\nu}{\varepsilon(\rho+(1-\alpha)\nu+\lambda)}\right)\hat{theta}^p. \tag{66}$$

Note that the coefficient on the right-hand side is increasing in λ. Moreover, the rate of decay β^f that characterizes the homogeneous solution satisfies

$$0 = \beta^f - \frac{\omega^*-1}{\varepsilon} - \frac{1}{\rho+\beta^f}\frac{\omega^*}{\varepsilon\chi n^*} - \frac{\rho+\nu}{\varepsilon(\rho+(1-\alpha)\nu+\beta^f)}\Bigg).$$

Therefore, if $\lambda < \beta^f$, the coefficient on the right-hand side of (66) is negative, so $\hat{k}^{CB}$ and $\hat{theta}^p$ have opposite signs. Note, furthermore, that $\hat{y}^p$ and $\hat{py}^p$ have the same sign as $\hat{\theta}^p$.

Suppose $\lambda < \beta^f$ and $\nu^k > 0$. Then for large t,

$$\hat{\theta}_t = e^{-\beta_f t}\hat{\theta}^f + e^{-\lambda t}\hat{\theta}^p \to e^{-\lambda t}\hat{\theta}^p < 0.$$

Therefore, $\hat{y}_t$ and $\hat{py}_t$ are also negative for large enough t, since they have the same sign as $\nu^k + \hat{\theta}^p$, which satisfies

$$\left(\frac{\omega^*-1}{\varepsilon}+\frac{1}{\rho+\lambda}\frac{\omega^*}{n^*\chi\varepsilon}+\frac{\rho+\nu}{\varepsilon(\rho+(1-\alpha)\nu+\lambda)}\right)(\nu^k+\hat{\theta}^p) = \lambda\hat{\theta}^p,$$

so the left-hand side is negative.

Proofs of Propositions 8 and 9. The calculations above completely characterize the impulse response to an asset purchase shock, proving both results. □

D Optimal policy

D.1 Preliminaries

I begin by demonstrating that the welfare function can be written solely in terms of output gaps $\tilde{y}_t$ and the price-output ratio py_t. The household's flow payoff is

$$
\begin{aligned}
U_t &= \log C_t - \ell_t \\
&= \log A_t + \log y_t + \log(1 - \frac{x_t k_t}{y_t}) - \left(\frac{y_t}{k_t^\alpha}\right)^{\frac{1}{1-\alpha}} \\
&= \log A_t + \alpha \log k_t + \log \tilde{y}_t + \log(1 - \nu py_t) - \tilde{y}_t^{\frac{1}{1-\alpha}}
\end{aligned}
$$

Next, observe that the evolution of the capital stock (3) can be written as

$$
\begin{aligned}
d\log k_t &= (\phi(x_t) - \delta)dt \\
&= \nu(\log \nu - \delta + \log y_t + \log py_t - \log k_t) \\
&= \nu(\log \nu - \delta + \log \tilde{y}_t + \log py_t - (1-\alpha)\log k_t)
\end{aligned}
$$

This differential equation has the well-known solution

$$
\log k_t = e^{-(1-\alpha)\nu t} \log k_0 + \int_0^t e^{-(1-\alpha)\nu(t-s)} \nu(\log \nu - \delta + \log \tilde{y}_s + \log py_s) ds. \tag{67}
$$

Total welfare can then be written as

$$
\begin{aligned}
W &= \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} U_t dt \right] \\
&= \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\log A_t + \alpha \log k_t + \log \tilde{y}_t + \log(1 - \nu py_t) - \tilde{y}_t^{\frac{1}{1-\alpha}} \right) \right] \\
&= W_0 + \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} (\log \tilde{y}_t + \log(1 - \nu py_t)) - \tilde{y}_t^{\frac{1}{1-\alpha}} + \alpha\nu \left(\int_0^t e^{-(1-\alpha)\nu s} \log \tilde{y}_s + \log py_s ds \right) dt \right] \\
&= W_0 + \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left((1 + \frac{\alpha\nu}{\rho + (1-\alpha)\nu} \log \tilde{y}_t - \tilde{y}_t^{\frac{1}{1-\alpha}} \right) dt \right] \\
&+ \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\log(1 - \nu py_t) + \frac{\alpha\nu}{\rho + (1-\alpha)\nu} \log py_t \right) dt \right].
\end{aligned}
$$

where W_0 is an exogenous constant that does not depend on $(\tilde{y}_t, py_t)$.

Henceforth, I will write the welfare function as

$$W = \mathbb{E}_0\left[\int_0^\infty u(\tilde{y}_t, py_t)dt\right], \tag{68}$$

where

$$u(\tilde{y}, py) = (1 + \frac{\alpha\nu}{\rho + (1-\alpha)\nu})\log\tilde{y} - \tilde{y}^{\frac{1}{1-\alpha}} + \log(1 - \nu py_t) + \frac{\alpha\nu}{\rho + (1-\alpha)\nu}\log py_t.$$

Proof of Proposition 10. The problem of finding an optimal allocation reduces to choosing $(\tilde{y}_t, py_t)$ to maximize (68). There is a simple closed-form solution to this problem:

$$\tilde{y}_t = \left(\frac{(1-\alpha)(\rho+\nu)}{\rho + (1-\alpha)\nu}\right)^{\frac{1}{1-\alpha}},$$

$$py_t = \frac{\alpha}{\rho+\nu}.$$

□

Next, I show that the first-best can be attained if and only if the central bank uses both interest rate policy and asset purchases.

Proof of Proposition 11. Throughout the proof, I define the first-best level of output as

$$Y_t^* = K_t^\alpha(A_t\frac{(1-\alpha)(\rho+\nu)}{\rho+(1-\alpha)\nu})^{1-\alpha}.$$

I guess and verify that an optimal policy sets (i_t, K_t^{CB}) so that

$$r_t = r_t^* \equiv \rho + \alpha\mu_{Kt} + (1-\alpha)g - \sigma^2,\ \eta_t = 0$$

for all t, where

$$\mu_{Kt} = \nu\left(\log\frac{\alpha}{\rho+\nu} + \alpha\log K_t + (1-\alpha)(\log A_t + \log\ell^*) - \delta\right).$$

I show that there exists an equilibrium with $Y_t = Y_t^*$ and $\frac{q_t K_t}{Y_t} = \frac{\alpha}{\rho+\nu}$ for all t.

To guarantee $\eta_t = 0$ for all t, the central bank needs to set K_t^{CB} so that

$$\frac{K_t^{CB}}{K_t} = 1 - \sum_{i=1}^{I} \theta_{it}\omega_i^* \quad \text{for all} \quad t. \tag{69}$$

If $\eta_t = 0$ for all t, then $\omega_{it} = \omega_i^*$, so the market-clearing condition (13) implies that $\eta_t = 0$ is indeed an equilibrium. To guarantee $r_t = r^*$ for all t, the central bank needs to set

$$i_t = r^* + \pi_t \quad \text{for all} \quad t. \tag{70}$$

If $r_t = r^*$ for all t, then it immediately follows that consumption obeys

$$\frac{dC_t}{C_t} = gdt + \sigma dZ_t.$$

If $\eta_t = 0$ for all t as well, then asset prices are priced according to the household's SDF:

$$(r^* + \sigma_{Ct}\sigma_{qt})q_t K_t = \alpha Y_t - \nu q_t K_t + \nu(\log q_t - \delta)q_t K_t + \mu_{qt} q_t K_t.$$

It immediately follows that the level of investment is optimal from the household's perspective (given consumption C_t and aggregate output Y_t). Thus, the price-output ratio is at its first-best level, $py_t = \frac{\alpha}{\rho+\nu}$, so $q_t = \frac{\alpha}{\rho+\nu}\frac{Y_t}{K_t}$.

The capital stock evolves according to

$$d\log K_t = \nu\Bigg(\log\big(\frac{\alpha}{\rho+\nu}\frac{Y_t}{K_t}\big) - \delta\Bigg)dt,$$

so under the conjecture that $Y_t = Y_t^*$, the first-best level of output grows according to

$$d\log K_t = \nu\Bigg(\log\frac{\alpha}{\rho+\nu} + \alpha\log K_t + (1-\alpha)(\log A_t + \log \ell^*) - \delta\Bigg)dt \equiv \mu_{Kt}dt.$$

The first-best level of output then evolves according to

$$\begin{aligned} d\log Y_t^* &= \alpha d\log K_t + (1-\alpha)d\log A_t \\ &= (\mu_{Kt} + g - \frac{1}{2}\sigma^2)dt + \sigma dZ_t \end{aligned}$$

Note that under the conjectured equilibrium, $d\log C_t = d\log Y_t^*$, since labor $\ell_t = \ell^*$ is constant. Then, as long as

$$r_t = \alpha\mu_{Kt} + (1-\alpha)g - \sigma^2,$$

there exists an equilibrium in which $Y_t = Y_t^*$ and $\frac{q_t K_t}{Y_t} = \frac{\alpha}{\rho+\nu}$.

Next, I show that generically, it is impossible to implement the first-best using interest rate policy only. This result follows immediately from the fact that under the first-best, it must be that $r_t = r_t^*$ and $\eta_t = 0$ for all t. However, above I have shown that if i_t is set so that $r_t = r_t^*$, then generically it must be that $K_t^{CB} \neq 0$ to guarantee that $\eta_t = 0$ for all t. $\square$

D.2 The approximate policy problem and its solution

I derive the optimal policy by solving the problem in discrete time, with time intervals of length Δ, and taking the limit as $\Delta \to 0$. First, note that there exist constants g_y, g_{py}, g_k such that the planner's flow payoff at time t is approximately

$$\tilde{u}(\hat{\tilde{y}}_t, \hat{py}_t, \hat{k}_t^{CB}) = \tilde{u}_0 - \frac{1}{2}\Big(\lambda_y \hat{\tilde{y}}_t^2 + \lambda_{py}\hat{py}_t^2 + \lambda_k (\hat{k}_t^{CB})^2\Big),$$

where $\tilde{u}_0$ denotes $\tilde{u}(0,0,0)$. The planner's problem with a time interval can then be written as

$$\max_{\hat{\tilde{y}}_t, \hat{py}_t, \hat{\theta}_t, \hat{\eta}_t, \hat{r}_t} -\mathbb{E}_0\left[\int_0^\infty \frac{1}{2}\Big(\lambda_y \hat{\tilde{y}}_t^2 + \lambda_{py}\hat{py}_t^2 + \lambda_k(\hat{k}^{CB})^2\Big)\right] \quad \text{s.t.} \quad \begin{aligned} \hat{\eta}_t &= -(\rho + (1-\alpha)\nu)\hat{py}_t + \dot{\hat{py}}_t \\ \hat{r}_t &= \dot{\hat{\tilde{y}}}_t + \dot{\hat{py}}_t \\ \dot{\hat{\theta}}_t &= (\omega^* - 1)\hat{\eta}_t - (\rho+\nu)\hat{py}_t. \end{aligned} \tag{71}$$

I use the recursive contracting techniques of Marcet and Marimon (2019) to find the optimal policy. The recursive policy problem with a time interval of length Δ is

$$\begin{aligned} W(\mu, \zeta, \hat{\theta}) = \min_{\gamma,\zeta} \max_{\hat{\tilde{y}}, \hat{py}, \hat{k}^{CB}, \hat{r}} & -\frac{1}{2}\Big(\lambda_y \hat{\tilde{y}}^2 + \lambda_{py}\hat{py}^2 + \lambda_k(\hat{k}^{CB})^2\Big)\Delta + (1+\rho\Delta)\zeta\hat{\tilde{y}} - \xi(\hat{\tilde{y}} + \hat{r}\Delta) \\ & - \gamma\Big(-\frac{\hat{\theta} + \hat{k}^{CB}}{\varepsilon}\Delta - (1 + (\rho + (1-\alpha)\nu)\Delta)\hat{py}\Big) + (1+\rho\Delta)\mu\hat{py} \\ & + (1-\rho\Delta)\,\mathbb{E}\left[W\big(\gamma, \xi, \hat{\theta} - (\omega^* - 1)\frac{\hat{\theta} + \hat{k}^{CB}}{\varepsilon}\Delta - \hat{r}\Delta - (\rho+\nu)\hat{py}\Delta\big)\right]. \end{aligned}$$

The envelope theorem implies that

$$W_{\hat{\theta},t} = (1-\rho\Delta)(1 - \frac{\omega^* - 1}{\varepsilon}\Delta)\,\mathbb{E}_t[W_{\hat{\theta},t+\Delta}] + \frac{\mu_{t+\Delta}}{\varepsilon}\Delta.$$

The first-order conditions are

$$(\hat{k}_t^{CB}) : 0 = -\lambda_k k_t^{CB}\Delta + \frac{\mu_{t+\Delta}}{\varepsilon}\Delta - (1-\rho\Delta)\frac{\omega^* - 1}{\varepsilon}\Delta\,\mathbb{E}[W_{\hat{\theta},t+\Delta}]$$
$$(\hat{r}_t) : 0 = -\zeta_{t+\Delta}\Delta - (1-\rho\Delta)\Delta\,\mathbb{E}_t[W_{\hat{\theta},t+\Delta}]$$
$$(\hat{\tilde{y}}_t) : 0 = -\lambda_y \hat{\tilde{y}}_t\Delta + (1+\rho\Delta)\zeta_t - \zeta_{t+\Delta}$$
$$(\hat{py}_t) : 0 = -\lambda_{py}\hat{py}_t\Delta + (1+(\rho+(1-\alpha)\nu)\Delta)\mu_{t+\Delta} - (1+\rho\Delta)\mu_t - (1-\rho\Delta)(\rho+\nu)\Delta\,\mathbb{E}_t[W_{\hat{\theta},t+\Delta}]$$

In the limit $\Delta \to 0$, these optimality conditions imply $\zeta_t = \mathbb{E}_{t-\Delta}[W_{\hat{\theta}t}]$ and

$$\rho\zeta_t = \frac{1}{\varepsilon}\mu_t - \frac{\omega^* - 1}{\varepsilon}\zeta_t + \dot{\zeta}_t, \tag{72}$$

$$-\frac{\hat{\theta}_t + \hat{k}_t^{CB}}{\varepsilon} = (\rho + (1-\alpha)\nu)\hat{py}_t - \dot{\hat{py}}_t. \tag{73}$$

$$\dot{\theta}_t = -\frac{(\omega^* - 1)}{\varepsilon}(\hat{\theta}_t + \hat{k}_t^{CB}) - \dot{\hat{\tilde{y}}}_t - (\rho+\nu)\hat{py}_t \tag{74}$$

$$\lambda_{py}\hat{py}_t = (1-\alpha)\nu\mu_t + \dot{\mu}_t - (\rho+\nu)\zeta_t. \tag{75}$$

$$\lambda_y\hat{\tilde{y}}_t = \lambda_k\hat{k}_t^{CB} = \frac{1}{\varepsilon}\mu_t - \frac{\omega^* - 1}{\varepsilon}\zeta_t \tag{76}$$

Proof of Proposition 12. Inspecting (76), it is immediate that $\hat{\tilde{y}}_t = \frac{\lambda_k}{\lambda_y}\hat{k}_t^{CB}$. □

Let $\mathbf{a}_t = (\mu_t, \zeta_t, \theta_t)'$. Plugging (75) and (76) to (72)-(74),

$$M_2\ddot{\mathbf{a}}_t + M_1\dot{\mathbf{a}}_t + M_0\mathbf{a}_t = 0, \tag{77}$$

where

$$M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, M_1 = \begin{pmatrix} 0 & 1 & 0 \\ (1-\alpha)\nu - \frac{\rho+(1-\alpha)\nu}{\lambda_{py}} & -0 & 0 \\ -\frac{1}{\varepsilon\lambda_y} + \frac{\rho+\nu}{\lambda_{py}} & -\frac{\omega^*-1}{\varepsilon\lambda_y} & 1 \end{pmatrix},$$

$$M_0 = \begin{pmatrix} \frac{1}{\varepsilon} & -\rho - \frac{\omega^*-1}{\varepsilon} & 0 \\ -\frac{1}{\varepsilon^2\lambda_y} - \frac{(1-\alpha)\nu(\rho+(1-\alpha)\nu)}{\lambda_{py}} & \frac{\omega^*-1}{\varepsilon^2} + \frac{(\rho+(1-\alpha)\nu)(\rho+\nu)}{\lambda_{py}} & -\frac{1}{\varepsilon} \\ \frac{(\rho+\nu)(1-\alpha)\nu}{\lambda_{py}} + \frac{\omega^*-1}{\varepsilon^2\lambda_k} & (\frac{\omega^*-1}{\varepsilon})^2\frac{1}{\lambda_k} + \frac{(\rho+\nu)^2}{\lambda_{py}} & \frac{\omega^*-1}{\varepsilon} \end{pmatrix}.$$

This is a second-order constant-coefficient ordinary differential equation in t. It has a solution of the form

$$\begin{pmatrix} \mu_t \\ \zeta_t \\ \theta_t \end{pmatrix} = e^{-Zt}\begin{pmatrix} \mu_0 \\ \zeta_0 \\ \theta_0 \end{pmatrix} \tag{78}$$

for some matrix Z.

Proof of Propositions (13) and (14). Note that (76) and (75), along with the solution (78) imply that $\hat{k}_t^{CB}, \hat{py}_t$ (as well as their time derivatives) are linear combinations of $(\zeta_t, \hat{\theta}_t)$ as defined above. Then both can in fact be written as linear combinations of $(\hat{k}_t^{CB}, \hat{\theta}_t)$. $\square$