A Model of Collateralized Lending Chains

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Abstract

Collateral is often used to secure promises among multiple parties in debt markets. I present a model in which agents with heterogeneous beliefs borrow by using a physical asset and the liabilities of other agents as collateral. In equilibrium, a chain of lending emerges: each agent lends to the next-most optimistic agent and borrows from the next-most pessimistic agent. This leads to a tranched payoff structure in which each agent takes losses only after their debtors are wiped out. Intermediation allows optimists to lever up while pessimists invest in safe assets. In extensions of the benchmark model, I examine the implications of this arrangement for financial stability. I show that (1) chains of lending lead to contagion in margin-driven crashes along the chain of lending and across asset classes, (2) leverage and private-label safe asset production comove positively, and (3) financial innovation leads both borrowers and intermediaries to take riskier positions.

1 Introduction

Recent innovations have led to the use of collateral to secure promises among multiple parties in complex financial arrangements. For example, a collateralized mortgage obligation (CMO) is a debt contract backed by mortgages, which in turn are debt contracts backed by houses. This type of arrangement is termed "pyramiding" by Geanakoplos (1997). In repo markets, lenders can finance themselves by rehypothecating the borrowed security, thereby spreading the collateral further. Finally, tranched securities allow for the collateral to be split between multiple parties at once: the owners of a given tranche are entitled to the residual cash flows from the collateral whenever the tranche begins to take losses. During the financial crisis of 2007-2009, margins on collateralized lending rose dramatically and markets involving these types of complex arrangements collapsed. In order to address the emergence of these arrangements and the the consequences for financial stability, this paper studies the implications of collateralized lending among multiple parties.

I present a model of lending chains in which an asset is sold to N types of agents who are permitted to use the asset or debt contracts as collateral. Gains from trade stem from differences in beliefs, as in Geanakoplos (2010) or Simsek (2013). In this model, the most optimistic agents always purchase the asset, and other agents engage in a chain of borrowing and lending: each type lends to the next most optimistic type and borrows from the next most pessimistic type, using their long position in debt contracts as collateral. This financing structure leads to an endogenous tranching of the collateral. Agents walk away from their obligations whenever the value of the assets used to collateralize a loan falls below the repayment due on that loan. When an agent's payoffs are wiped out, the asset is passed on to the next agent in the lending chain. In fact, in this model lending chains lead to the exact same outcome as direct tranching of the asset, which suggests that heterogeneity may lead to tranched payoffs independently of the precise institutional details of financial markets.

The first contribution of this work is to outline how belief heterogeneity can lead to endogenous intermediation. In the model, optimists receive payoffs only in good states because they are relatively more optimistic about those states. Optimists therefore take risky loans from other agents. Intermediate types are relatively more optimistic about intermediate states, so to receive payoffs in these states, they must default only in bad states. To accomplish this, these types act as intermediaries: they lend to optimistic types to secure safe payoffs in good states and fund themselves by borrowing from pessimistic agents who demand relatively safe assets. As discussed in the literature review (Section 1.1), belief heterogeneity is a novel explanation of intermediation and is to my knowledge the only rationalization of *endogenous* chains of lending in the literature.

In order to understand the role of intermediation in determining margins and asset prices simultaneously, it is critical to have a model in which intermediation is endogenous. I study these questions by characterizing the dependence of leverage, safe asset issuance, and asset prices on intermediaries' net worth and the length of lending chains. Long intermediation chains result in the alignment of agents' portfolios with their risk preferences. When an intermediary's capitalization increases, all downstream borrowers are able to lever up and take riskier loans at lower interest rates. By contrast, all upstream lenders benefit from the additional cash buffer provided by the intermediary and make safer loans. The role of intermediation is thus to simultaneously increase leverage and create safe assets. This model illustrates how the rise of leverage and the supply of safe assets and the subsequent fall in these quantities after the most recent financial crisis can be seen as a reduction in the financial sector's intermediation capacity.

Agents' ability to reuse debt contracts as collateral and create arbitrarily long lending chains in the model yields additional theoretical insights. I show that the reusability of collateral reduces the problem of pricing the underlying asset to pricing state-specific contracts that pay off 1 if the asset's payoff (the realized state of nature s) is above a certain cutoff $\hat{s}$

and zero otherwise. Agents can produce these securities by lending against a certain promised repayment $\hat{s}$ and using that contract as collateral to promise a slightly smaller repayment $\hat{s} - \epsilon$ to a lender. The fact that the asset's payoff can be deconstructed in this way makes my setting analogous to an Arrow-Debreu economy in which the asset's price is just a sum of the prices of assets that pay off 1 in each possible future state. In my model, the analogue of the Arrow-Debreu security that pays 1 in state s is the portfolio that pays 1 in all states greater than s. The link between re-pledgeability of collateral and the decomposition of the underlying asset's price into the prices of state-specific securities is also new to the literature.

I apply and extend the model to study several features of markets that feature chains of collateralized lending. First, I ask whether the model can produce contagion in which margins spike in several markets simultaneously. Geanakoplos (2010) faults a "double leverage cycle" in which lending conditions in housing markets and repo markets for mortgage-backed securities fed back on each other for much of the turmoil in financial markets during the crisis of 2007-2009. Gorton and Metrick (2012) show that margins in the bilateral repo market spiked in essentially all risky asset classes. The model of collateralized lending chains can generate both contagion *along* the chain of lending (i.e., double leverage cycles) and *across* asset classes, even when the payoffs of those assets are uncorrelated. The simple intuition behind my results is that when bad news scares some agents and tightens lending conditions in one market, intermediaries become constrained in their ability to borrow and therefore lend less against any security. This explanation of contagion in margins as a pure wealth effect is more parsimonious than others offered in the literature, as discussed in the literature review.

Second, I show that an extension of the model with an investment technology can produce cycles in which leverage, safe asset production, asset prices, and investment all comove positively with intermediary net worth. Empirically, Fostel and Geanakoplos (2016) document that leverage and investment in housing markets tracked real estate prices during the period 2000-2009, and Lenel (2017) shows that production of private-label long-maturity safe assets reached record levels before the financial crisis and then collapsed after 2008. My model rationalizes these trends as an increase and subsequent decrease in the intermediary (or shadow banking) sector's ability to act as a cash buffer and protect risk-averse investors from crashes in mortgage-backed security prices while providing end-borrowers with low-margin loans. I present a three-period version of the model in which the intermediary's net worth fluctuates over time and show using numerical examples that even when intermediaries see the potential of a great buying opportunity after a crash, they still expose themselves to aggregate risk, so the empirical patterns listed above are replicated in the dynamic version of the model.

Third, I present an alternative model in which debt contracts cannot be used as collateral in order to understand how the transition to a world in which long chains of lending become possible affects risk-taking. I show that this specific type of financial innovation either increases the risk taken by agents who become intermediary lenders or increases the risk taken by end-borrowers who benefit from the additional leverage provided by levered intermediaries. Volatility in the economy with debt contracts as collateral is much higher because lenders can lever up and go bankrupt precisely in the states in which intermediation is needed most.

The paper is organized as follows. The rest of this section is dedicated to a review of the relevant literature. Section 2 presents the model and defines an equilibrium. In Section 3, I characterize and describe the properties of equilibria (including the endogenous tranched payoff structure). Section 4 studies the consequences of intermediation in detail. Section 5 discusses several applications of the model. I discuss the model's assumptions in Section 6. Section 7 concludes.

1.1 Literature Review

This paper is related to the literature on collateralized lending with heterogeneous beliefs, as in Geanakoplos (2003, 2009, 2010) and Simsek (2013). In particular, the assumptions on beliefs and the structure of the economy are very similar to those in Simsek (2013). This model departs from the models in those works in that it allows for the use of debt contracts as collateral. This assumption allows agents to act as intermediaries and gives rise to the endogenous multi-level tranching of the asset. Without allowing for the reuse of collateral, it would be impossible to analyze how intermediaries' beliefs and wealth affect asset prices and leverage. My results regarding how beliefs affect the equilibrium price are similar to those in Geanakoplos (2003, 2010) and Simsek (2013) in that I find increased disagreement can increase margins, thereby lowering the asset price. In this model, however, there is more than one margin because there is a chain of lending rather than a single debt contract, so my result has additional qualifications. Geerolf (2017) also shows that chains of lending can emerge in equilibrium, but that paper relies on dogmatic beliefs under which agents are completely certain that a given asset payoff will be realized. My paper is the first to illuminate the nature of endogenous chains of lending and show that the allocation with reusable collateral coincides with that in a dual Arrow-Debreu economy.

This paper is also a part of the theoretical literature on collateralized lending, securitization, and lending chains. Bottazzi, Luque, and Pscoa (2012) show the existence of equilibrium in an economy with rehypothecation of collateral and the liquidity premium associated with pledgeable assets. Muley (2016) studies the optimality of rehypothecation and securitization in an environment with limited commitment. Dang, Gorton, and Hölmstrom (2013) demonstrate that haircuts arise in order to solve an adverse selection problem faced by lenders and study how haircuts affect credit in lending chains. Other recent papers, such as Di Maggio and Tahbaz-Salehi (2015), Infante (2015), and Kahn and Park (2015) attempt to understand the implications of collateralized lending in a setting where credit relationships are exogenously given. In my model, by contrast, lending chains form endogenously as agents attempt to align their portfolios with their appetites for risk.

This literature on asset pricing with margin constraints is relevant to my paper as well. For example, Garleanu and Pedersen (2011) and Rytchkov (2014) model economies with credit constraints that affect asset prices. Unlike those papers, in my model credit constraints are endogenous and come from beliefs rather than being given exogenously or resulting from limited commitment. Brunnermeier and Pedersen (2009) also examines asset markets with endogenous margins and concludes, as I do, that spikes in margins can be contagious across asset classes, but their explanation of how margins are set in equilibrium is more involved and contagion does not reduce to a simple wealth effect, as in my model.

There is a large body of empirical work related to the recent financial crisis and the collapse of repo (i.e., collateralized lending) markets. Gorton and Metrick (2012) provides a timeline of the financial crisis and examines the increase in margins on asset-backed securities during that period. Krishnamurthy, Nagel, and Orlov (2014) argue that the run on aggregate repo borrowing was small but concentrated on a handful of systematically important borrowers. Copeland, Martin, and Walker (2014) provide some evidence from the triparty repo market to support this hypothesis. Shin (2009) studies the effects of deleveraging in a complex, interconnected financial system. The consequences of a reduction in "collateral velocity" are examined in Singh (2011). Reinhart and Rogoff (2008) emphasize the role of mistaken beliefs in serious financial crises.

2 Model

2.1 Environment

The economy exists for two periods, $t = 0$ and $t = 1$. There is a single consumption good, which is referred to as a dollar. There is a continuum of each of N types of agents who invest their endowments at $t = 0$ in order to consume at $t = 1$ and a continuum of agents who sell an asset a in order to consume at $t = 0$. The only source of uncertainty is the terminal payoff of the asset: there is a continuum of states $s \in [0, \overline{s}] = S$ that index the payoff of the asset in terms of the consumption good at $t = 1$.

Agents differ in their degree of optimism about the asset's payoff, meaning each type n has a different subjective probability measure on $[0, \overline{s}]$ defined by a density function $f_n(s)$.

The following assumption formalizes the notion of optimism:

Assumption 1. *The types of agents $n \in \{1, 2, \ldots, N\}$ are ordered by their optimism, with type $n = 1$ being the most optimistic and $n = N$ being the most pessimistic, in the sense that the hazard rate inequality*

$$\frac{f_n(s)}{1 - F_n(s)} < \frac{f_{n+1}(s)}{1 - F_{n+1}(s)}$$

is satisfied for $n = 1, \ldots, N-1$ (where F_n is the CDF corresponding to f_n).

This property implies that F_n first-order stochastically dominates F_{n+1}, but it is weaker than the monotone likelihood ratio property.

All agents invest in order to maximize expected consumption $E_n[c]$ at $t = 1$. Agents of type n can invest their endowments w_n at $t = 0$ in one of three ways: they may purchase the asset, lend to other agents, or hold cash, which yields a safe return equal to one dollar at $t = 1$. In this economy agents are permitted to default, so borrowers must use collateral in order to provide agents an incentive to lend. That is, in order to agree to pay a lender θ units of the consumption good at $t = 1$, a borrower must set aside one unit of some asset (which could be the asset a or a debt contract) as collateral in case of default. In principle, the amount of money raised by the borrower could depend on the collateral.

Let

$$A : \{\phi(s) : \phi : S \to \mathbb{R}_+ \textit{ measurable and bounded}\}$$

be the set of assets. Formally, a debt contract is defined as a pair (θ, ϕ), where θ is the amount the borrower agrees to repay at $t = 1$ and $\phi \in A$ is the asset used as collateral. The debt contract is itself an asset with payoff $\min\{\theta, \phi(s)\}$, since the borrower chooses to default whenever the collateral's payoff is less than the amount owed. In this setting, the space of assets is restricted to $\tilde{A} = \underset{k}{\cup} A^k$, where the sets A^k are defined inductively as

$$A^0 = a, \;\; A^k = \{\min\{\theta, \phi(s)\} : \theta \in \mathbb{R}_+, \phi \in A^{k-1}\}$$

The set A^1 consists of debt contracts that use a unit of the asset a as collateral. The set A^2 consists of debt contracts that use debt contracts in A^1 as collateral, and so forth. The set $\tilde{A}$ is therefore the set of all assets that can be constructed using only a and collateralized debt contracts.

Proposition 2.1. *The set of assets is $\tilde{A} = A^0 \cup A^1$.*

All proofs are relegated to the Appendix. This proposition implies that no matter what collateral is used, the payoff of a debt contract in which the borrower agrees to pay θ at $t = 1$

is $\min\{\theta, s\}$ (since the lender would never accept collateral that pays less than θ in every state). The space of assets therefore consists of functions of the type $\min\{\theta, s\}$ with $\theta \in S$. In equilibrium, then, the amount of money a borrower can raise by promising θ at $t = 1$ will be independent of the asset used as collateral so long as that asset does not deliver less than θ in every state. Such a contract will henceforth be referred to as a loan of riskiness θ.

Note that whenever a borrower defaults on a loan of riskiness θ in a state $s < \theta$, the lender's payoff at $t = 1$ is s. It is as if lenders receive the asset itself in the event of default even if the collateral was a loan of riskiness $\theta' > \theta$. Thus there is a sense in which using a debt contract as collateral is equivalent to rehypothecating the asset, so in this model lending chains resemble an arrangement in which lenders take possession of collateral and are permitted to reuse it, as in a repo market.

2.2 Optimization Problem

Let p be the price of the asset and $q(\theta)$ be the amount borrowed at $t = 0$ when taking a loan of riskiness θ. The problem of an agent of type n is

$$\max_{a,\mu_+,\mu_-,c} aE_n[s] + c + \int E_n[\min\{\theta, s\}]d\mu_+(\theta) - \int E_n[\min\{\theta, s\}]d\mu_-(\theta)$$

$$s.t.\ w_n = pa + c + \int q(\theta)d\mu_+(\theta) - \int q(\theta)d\mu_-(\theta)$$

$$a + \int_{\theta \geq \hat{\theta}} d\mu_+(\theta) \geq \int_{\theta \geq \hat{\theta}} d\mu_-(\theta)\ \forall \hat{\theta} \in S,\ \mu_+(\theta) \geq 0,\ \mu_-(\theta) \geq 0$$

That is, agents choose the quantity a of the asset to purchase, measures $\mu_+(\theta)$ and $\mu_-(\theta)$ corresponding to the amount of lending and borrowing they choose for each $\theta \in S$, respectively, and an amount c of cash to hold. Their choices are subject to their budget constraints (the second line) and collateral constraints (the third line). Every contract in which the agent agrees to pay θ at $t = 1$ must be backed by one unit of the asset or a lending contract in which the agent receives $\theta' \geq \theta$. Absence of arbitrage will imply $q(\overline{s}) \leq p$.

Consider the alternative formulation

$$\max_{\mu,c} \int_{\theta' \leq \theta} E_n[\min\{\theta, s\} - \min\{\theta', s\}]d\mu(\theta, \theta') + c$$

$$s.t.\ w_n = \int_{\theta' \leq \theta} (q(\theta) - q(\theta'))d\mu(\theta, \theta') + c,\ \ \mu(\theta, \theta') \geq 0$$

It is possible to show the constraint set in this problem is the same as that in the original problem (after a change of variables). The two problems are thus equivalent. Henceforth this formulation will be used for analytical tractability. Note that in this problem, agents choose a measure μ defined on the space

$$\hat{A} = \{\phi_1(s) - \phi_2(s) : \phi_1(s), \phi_2(s) \in \tilde{A}, \phi_2 \leq \phi_1\} = \{\min\{\theta, s\} - \min\{\theta', s\} : \theta' \leq \theta\}$$

In this problem, agents choose $\mu(\theta, \theta')$, which represents lending funded by borrowing in which the agent receives θ and pays θ' at $t = 1$ or asset purchases funded by borrowing when $\theta = \bar{s}$. This problem is isomorphic to a portfolio choice problem in which agents purchase assets (θ, θ') with expected payoffs $\pi(\theta, \theta') = E_n[\min\{\theta, s\} - \min\{\theta', s\}]$ and prices $\tilde{q}(\theta, \theta') = q(\theta) - q(\theta')$. Clearly, due to the linearity of the objective function, collateralizing a loan of riskiness θ' with a loan of riskiness θ is optimal for type n agents only if

$$\frac{E_n[\min\{\theta, s\} - \min\{\theta', s\}]}{q(\theta) - q(\theta')} \geq \max\left\{1, \max_{(\tilde{\theta}, \tilde{\theta}')} \frac{E_n[\min\{\tilde{\theta}, s\} - \min\{\tilde{\theta}', s\}]}{q(\tilde{\theta}) - q(\tilde{\theta}')}\right\} \equiv r_n$$

Let $\Theta^* = \{(\theta, \theta') : \frac{E_n[\min\{\theta,s\} - \min\{\theta',s\}]}{q(\theta) - q(\theta')} = r_n\}$. A choice of measure μ is optimal if and only if $\mu(\hat{A} \backslash \Theta^*) = 0$.

2.3 Equilibrium

In equilibrium, agents will maximize their objective functions and asset markets will clear. The following definition formalizes the notion of general equilibrium in this economy:

Definition 2.2. *A general equilibrium of this economy consists of measures* $(\mu_1, \mu_2, \ldots, \mu_N)$, *cash holdings* $(c_1, \ldots, c_N)$ *and prices* $q(\theta) : S \to \mathbb{R}_+$ *such that*

- *Taking* $q(\theta)$ *as given,* (μ_n, c_n) *solves agent* n*'s optimization problem for each* $n \in \{1, 2, \ldots, N\}$.
- *Debt markets and the market for the asset* a *clear, meaning*

$$\sum_{i=1}^{N} \int_{\theta' \leq \theta} (\min\{\theta, s\} - \min\{\theta', s\}) d\mu_n(\theta, \theta') = s \;\forall s \in S$$

The market clearing condition merits further explanation. The condition says that state by state, the sum of agents' payoffs resulting from borrowing, lending, and direct purchases of the asset is equal to the asset's payoff. This embeds the assumption that agents who consume at $t = 0$ supply one unit of the asset inelastically and the observation that in equilibrium, if debt markets clear, the amount of gross debt outstanding has no effect on the sum of agents' payoffs. One agent's payoff from lending at $t = 1$ is another agent's repayment.

3 Properties of Equilibrium

Given the need to compute the price function $q(\theta)$ for each $\theta \in S$, it may seem a daunting task to find the set of equilibria. In this section it will be shown that equilibrium allocations correspond to constrained social optima. The solution to the social planner's problem will shed light on the form of the price function $q(\theta)$ and the properties of equilibrium allocations.

3.1 Social Planner's Problem

Consider the problem of a benevolent social planner who partitions asset a's payoff among N agents using assets with payoffs of the form $E_n[\min\{\theta, s\} - \min\{\theta', s\}]$ and places weight λ_n on the subjective *ex-ante* utility of type n agents. The planner's problem is

$$\max_{\mu_1,\ldots,\mu_N} \sum_{n=1}^{N} \lambda_n \int_{\theta' \leq \theta} E_n[\min\{\theta, s\} - \min\{\theta', s\}] d\mu_n(\theta, \theta')$$

$$s.t. \sum_{n=1}^{N} \int_{\theta' \leq \theta} (\min\{\theta, s\} - \min\{\theta', s\}) d\mu_n(\theta, \theta') = s \ \forall s \in S, \ \mu_n \geq 0$$

Note that the constraints on the measures μ_n are exactly the general equilibrium market clearing conditions.

The social planner's problem is thus to optimally assign assets to agents in such a way that the asset market clearing conditions are replicated. However, in the social planner's problem the constraints should not be interpreted as having anything to do with market clearing, as there is no sense in which agents borrow from or lend to each other. Rather, these conditions guarantee that the solution represents some partition of the payoff of one unit of the asset a into payoffs of the form $\min\{\theta, s\} - \min\{\theta', s\}$. These payoffs already resemble those of tranched securities in that each such asset's payoff is constant over $s \geq \theta$, declines over $\theta > s \geq \theta'$, and is zero below θ'. It will soon be shown that each agent's payoff will also take this form at an optimum.

It is optimal for the social planner to assign an asset (θ, θ') to agent n only if

$$\lambda_n E_n[\min\{\theta, s\} - \min\{\theta', s\}] = \max_m \lambda_m E_m[\min\{\theta, s\} - \min\{\theta', s\}]$$

In order to determine when this condition holds, it will be useful to determine for which pairs (θ, θ') the inequality $\lambda_n E_n[\min\{\theta, s\} - \min\{\theta', s\}] > \lambda_m E_m[\min\{\theta, s\} - \min\{\theta', s\}]$ holds (for fixed $n < m$). The following lemma definitively answers this question.

Lemma 3.1. *Fix $n < m$. The following three statements hold:*

1. *There exists a weakly decreasing function $\overline{g}_{nm}(\theta) : S \to S$ such that $\lambda_n E_n[\min\{\theta', s\} - \min\{\theta, s\}] > \lambda_m E_m[\min\{\theta', s\} - \min\{\theta, s\}]$ iff $\theta' > \overline{g}_{nm}(\theta)$.*

2. *There exists a weakly decreasing function $\underline{g}_{nm}(\theta)$ such that $\lambda_n E_n[\min\{\theta, s\} - \min\{\theta', s\}] > \lambda_m E_m[\min\{\theta, s\} - \min\{\theta', s\}]$ iff $\theta' > \underline{g}_{nm}(\theta)$.*

3. *There exists $\theta^*_{nm} \in [0, \overline{s}]$ such that $\overline{g}_{nm}(\theta) = \theta$ if $\theta > \theta^*_{nm}$ and $\underline{g}_{nm}(\theta) = \theta$ if $\theta < \theta^*_{nm}$.*

Let $\theta^*_N = 0$, $\theta^*_1 = \max_{m>1} \theta^*_{1m}$, and inductively define $\theta^*_n = \min\{\max_{m>n} \theta^*_{nm}, \theta^*_{n-1}\}$ for $2 \leq n \leq N-1$. Then $\theta^*_N \leq \theta^*_{N-1} \leq \cdots \leq \theta^*_1$. Lemma 3.1 implies that

$$\theta^*_n \leq \theta' \leq \theta \leq \theta^*_{n-1} \Rightarrow \lambda_n E_n[\min\{\theta, s\} - \min\{\theta', s\}] = \max_m \lambda_m E_m[\min\{\theta, s\} - \min\{\theta', s\}]$$

Define $\Theta^*_n = \{(\theta, \theta') : [\theta', \theta] \subset [\theta^*_n, \theta^*_{n-1}]\}$. It is always suboptimal for the social planner to choose a μ_n such that $\mu_n(\hat{A} \backslash \Theta^*_n) > 0$. For example, for any feasible allocation such that $\mu_n((\theta^*_{n-2}, \theta^*_n)) = c > 0$, the social planner could do strictly better by setting $\mu_n((\theta^*_{n-1}, \theta^*_n)) = \mu_{n-1}((\theta^*_{n-2}, \theta^*_{n-1})) = c$, and the constraints would still be satisfied. Intuitively, the social planner's constraints allow assets (θ, θ') to be split up into any arbitrary collection of sub-assets, so the social planner always chooses to allocate assets (θ, θ') in such a way that each type n holds assets with $[\theta', \theta] \subset [\theta^*_n, \theta^*_{n-1}]$. It is precisely those assets that type n values most relative to other agents.

The social optimum is achieved by the allocation $\mu_n((\theta^*_{n-1}, \theta^*_n)) = 1, \mu_n(\theta, \theta') = 0$ for all other pairs (θ, θ'). Up to trivial changes in the measures μ_n, there is no other feasible allocation such that $\mu_n(\hat{A} \backslash \Theta^*_n) = 0$ for all n. In such an allocation, agents with $\theta^*_n = \theta^*_{n-1}$ receive nothing, so it is convenient to enumerate the agents who receive nonzero asset holdings by setting $k_1 = \operatorname{argmax}_n (\theta^*_n)(\mathbf{1}\{\theta^*_n < \overline{s}\})$, $k_j = \operatorname{argmax}_n (\theta^*_n)(\mathbf{1}\{\theta^*_n < \theta^*_{k_{j-1}}\})$. This allocation resembles the payoff structure of a tranched security. Agents of type k_1 receive payoffs that start falling in value immediately as s decreases below $\overline{s}$ and reach zero at $\theta^*_{k_1}$. Type k_j's payoffs are constant over $[\theta^*_{k_{j-1}}, \overline{s}]$ and fall to zero as s goes from $\theta^*_{k_{j-1}}$ to $\theta^*_{k_j}$.

Why should this be the constrained social optimum? At the margin, the planner decides whether to reduce the threshold $\underline{\theta}_n$ at which agent n's payoffs reach zero. An reduction in $\underline{\theta}_n$ reduces type n's perceived payoff by $1 - F_n(\underline{\theta}_n)$ since this reduction effectively gives type n an extra dollar in each state $s \geq \underline{\theta}_n$. Similarly, this reduction decreases type $n+1$'s payoff by $1 - F_{n+1}(\underline{\theta}_n)$, so the planner's decision depends on (1) the probabilities assigned to the tail event $s \geq \underline{\theta}_n$ by each type, and (2) the weights λ_n, λ_{n+1}. Given that n is more optimistic than $n+1$, even if $\lambda_{n+1} > \lambda_n$ there will be some $\theta^*_{n,n+1}$ such that the relative weight assigned to the tail event by type n outweighs the larger weight placed by the social planner on type $n+1$.

In general, this tranched allocation will not be an unconstrained social optimum in the following sense: if the planner were given access to a complete contingent set of securities rather than assets with payoffs $\min\{\theta, s\} - \min\{\theta', s\}$, the allocation would in general differ from the one obtained above. In fact, when agents' subjective probabilities satisfy the monotone likelihood ratio property (rather than the weaker hazard rate inequality), it can be shown that the planner chooses to give each type n the entire payoff of the asset over some range $s \in [\hat{\theta}_n, \hat{\theta}_{n-1}]$ and nothing in all other states. The optimality of tranching is therefore specific to an incomplete markets setting in which assets' payoffs resemble those that arise with collateralized lending.

3.2 Tranched Payoffs in Equilibrium

I now show that (1) the allocation that arises in any equilibrium corresponds to the allocation in some social optimum and (2) the equilibrium in this market exists and is unique.

Proposition 3.2. *Suppose $q(\theta), \{\mu_n\}_{n=1}^N, \{c_n\}_{n=1}^N$ is a competitive equilibrium. Then there exist weights $\{\lambda_n\}_{n=1}^N$ such that the payoffs of the constrained social optimum are achieved by $\{\mu_n\}_{n=1}^N$. Furthermore, there exist $\{\theta^*_n\}_{n=1}^N$ such that $q(\theta) - q(\theta') = \lambda_n E_n[\min\{\theta, s\} - \min\{\theta', s\}]$ whenever $[\theta', \theta] \subset [\theta^*_n, \theta^*_{n-1}]$ (where $\theta^*_0 \equiv \overline{s}$).*

Conversely, there is a unique mapping from the set $\Lambda = \{\{\lambda_n\}_{n=1}^N : \lambda_1 \leq \cdots \leq \lambda_N\}$ to the set of equilibrium allocations such that the resulting allocation coincides with the solution to the social planner's problem with weights λ.

Given that every equilibrium corresponds to a constrained social optimum, the previous characterization of constrained social optima carries through as a characterization of equilibrium allocations. That is, in any equilibrium, there are values $\{\theta^*_n\}_{n=1}^N$ such that $0 = \theta^*_N \leq \theta^*_{N-1} \leq \cdots \leq \theta^*_1 \leq \overline{s}$ and each type n owns assets with a total payoff of $\min\{\theta^*_{n-1}, s\} - \min\{\theta^*_n, s\}$. Tranched payoffs therefore always emerge in equilibrium: the most optimistic agent suffers losses whenever the asset's value falls below its maximum possible value, but the most pessimistic agent does not take losses until the asset's value falls below θ^*_{N-1}. These payoffs resemble those of CDO tranches. An example of the payoff structure is illustrated in Figure 1.

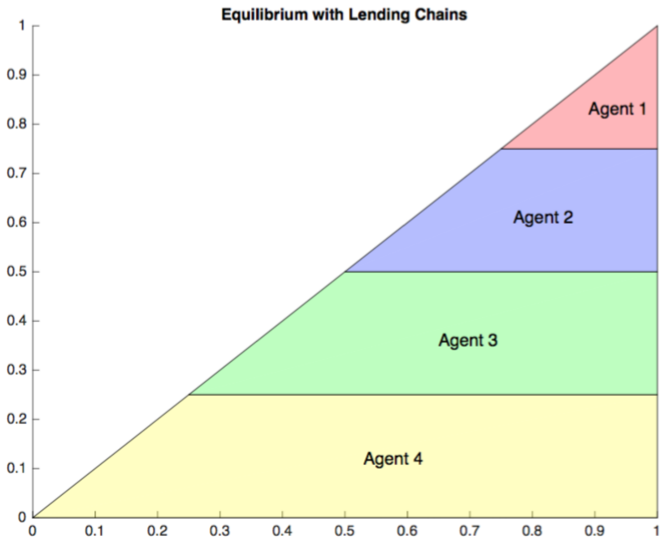

Figure 1: An example of the tranched payoff structure described in Propositions 3.2 and 3.3. Each agent's share of the asset's total payoff is shaded in a different color. An agent is wiped out when the realized state falls below a certain cutoff, at which point the next-most pessimistic type starts to take losses.

The mechanism that causes these payoffs to arise, however, is conceptually different. In equilibrium, type 1 buys the asset and uses it as collateral to borrow from type 2, promising to repay θ_1^* dollars at $t = 1$. Type 2 uses this debt contract as collateral to borrow from agent 3 and promises to repay θ_2^* dollars at $t = 1$. This continues all the way down to type N, who lends to type $N-1$ (without borrowing) in exchange for a promise to repay θ_{N-1}^* at t=1, taking $N-1$'s debt contract with $N-2$ as collateral.

Collateral in this type of arrangement is a contract backed by another contract, which in turn is backed by a third contract, and so forth, until the last contract, which is backed by a physical asset. This type of arrangement is another salient feature of housing markets: a CDO is a set of debt contracts backed by subprime mortgage-backed securities, which in turn are backed by pools of mortgages, which are contracts backed by houses. Chains of lending are also common in loan markets. Banks often lend to firms directly and then repo those loans to outside investors.

Agents find this behavior optimal because of the structure of their beliefs. The hazard-rate ordering of Assumption 1 ensures not only that low types are more optimistic than high types, but also that low types are relatively more optimistic conditional on sufficiently high realizations of s. To see this, note that the hazard rate assumption implies that $1 - F_i(s)$

decreases at a slower rate than $1 - F_j(s)$ whenever $i < j$. Consider the events $B_s = \{s' : s' \geq s\}$. The probability assigned to event B_s by a type k is $1 - F_k(s)$. If $i < j$, the ratio of the probability of B_s in type i's measure to that in type j's measure is $\frac{1-F_i(s)}{1-F_j(s)}$, which is increasing in s. In this sense, more optimistic types value risky payoffs relatively more than safe payoffs.

Intermediation emerges in this model precisely because intermediary types are most optimistic about intermediate states relative to optimists and pessimists. In order to construct a portfolio that pays off above some intermediate state, they lend to optimists, bear risk when optimists are wiped out, and borrow from pessimists. A common theme in the theoretical literature on intermediation is that types with preferences that are in some sense between those of other types will tend to take offsetting positions. For example, in Farboodi, Jarosch, and Shimer (2017) fast traders are less sensitive to the alignment of their preferences and asset holdings and act as intermediaries for slower traders. Atkeson, Eisfeldt, and Weill (2014) present a framework in which banks with intermediate risk exposures act as dealers for "customer" banks with extreme risk exposures. The emergence of lending chains in the presence of belief heterogeneity is a novel implication of this model.

Why is it that in this setup tranched payoffs arise in chains of lending? After all, in reality tranching arises when a single asset is used as collateral for many contracts, whereas in chains of lending contracts are used to back other contracts, with the physical asset serving as the collateral for the original contract. The answer lies in the fact that by borrowing and lending, agents in this model are able to synthesize assets whose payoffs are constant over some range of states and then decline to zero. When agents' opinions differ, they bet with each other on the tail probabilities of events. The most optimistic agents will naturally make the riskiest bets. At the interest rate that they are willing to borrow, the second most optimistic agents are happy to lend to them and take a safe payoff for draws in the extreme upper tail of the distribution of s. The social planner's solution indicates that whenever the set of available (synthetic) assets is $\hat{A}$, absent other frictions, agents will always make the exact same bets they do in this model.

Indeed, in this model there is an equivalence between chains of lending and the direct sale of tranched securities. Instead of assuming external agents sell an asset to agents who may borrow and lend with collateral, it could instead be assumed that these external agents offer a menu of assets with tranched payoffs $\min\{\theta, s\} - \min\{\theta', s\}$ in a competitive market at prices $\tilde{q}(\theta, \theta') = q(\theta) - q(\theta')$. In this case, clearly, agents would choose the same asset holdings, and equilibrium prices would be exactly the same. An alternative assumption is that some type n has the ability to create tranched securities and sell them to types $n+1, n+2, \ldots, N$. Types 1 through n would then take part in a chain of lending, and types $n+1$ through N would buy the tranches of the security sold by n, but again the equilibrium would be equivalent to the one obtained above. All results regarding chains of collateralized lending therefore also carry over to settings in which the asset is directly tranched.

Although Proposition 3.2 characterizes the equilibrium when it exists, it shows neither that an equilibrium always exists nor that it is unique. Fortunately, as Proposition 3.3 demonstrates, there always exists a unique equilibrium.

Proposition 3.3. *There exists a unique competitive equilibrium.*

3.3 Equilibrium Prices, Interest Rates, and Leverage

The equivalence between equilibrium allocations and constrained social optima yields interesting implications for prices and margins in the model. Consider an equilibrium with $0 = \theta_N^* < \theta_{N-1}^* < \cdots < \theta_1^* < \bar{s}$, and let $\{\lambda_n\}_{n=1}^N$ be the weights in the corresponding social planner's problem (assuming $\lambda_N = 1$). The weight placed on type n is the inverse of that type's return on wealth r_n (defined in Section 2.2). For the values θ_n^* to satisfy this property, it must be that $\theta_n^* = \theta_{n,n+1}^*$ (recalling that θ_{nm}^* is defined so that $\frac{1-F_n(\theta_{nm}^*)}{1-F_m(\theta_{nm}^*)} = \frac{\lambda_m}{\lambda_n}$). These assumptions may seem restrictive, but this is the only type of equilibrium in which all agents are involved in the chain of lending and the most pessimistic agent holds cash. That is, as long as each agent lends to the next most optimistic agent and borrows from the next most pessimistic agent, the equilibrium values of θ_n^* always take this form. Proposition 3.2 shows that the equilibrium always takes this form, so there is in fact no loss of generality. Whenever the most pessimistic agent holds cash, it must be that returns on wealth for type N are $r_N = 1$, since type N agents are content to receive zero net return on investment.

As shown in Proposition 3.2, for any $[\theta', \theta] \subset [\theta_{n,n+1}^*, \theta_{n-1,n}^*]$, the price function $q(\theta)$ satisfies

$$q(\theta) - q(\theta') = \lambda_n E_n[\min\{\theta, s\} - \min\{\theta', s\}] = \frac{1}{r_n} E_n[\min\{\theta, s\} - \min\{\theta', s\}] \qquad (1)$$

For any such pair (θ, θ'), $q(\theta) - q(\theta')$ is determined by the beliefs of type n. Relative to perceived returns on wealth r_n, type n has the highest valuation of the payoff $\min\{\theta, s\} - \min\{\theta', s\}$. Since $q(\theta) - q(\theta') > \frac{1}{r_n} E_n[\min\{\theta, s\} - \min\{\theta', s\}]$ for any $[\theta', \theta] \not\subset [\theta_{n,n+1}^*, \theta_{n-1,n}^*]$, type n will only borrow or lend quantities in the range $[\theta_{n,n+1}^*, \theta_{n-1,n}^*]$. All agents $n < N$ therefore use leverage when lending. In order to increase their returns on wealth, they borrow from more pessimistic agents who are willing to lend at favorable interest rates.

Given $q(0) = 0$ and $p = q(\overline{s})$, it follows that

$$p = \frac{1}{r_1} E_1[s - \min\{\theta^*_{1,2}, s\}] + \sum_{n>1} \frac{1}{r_n} E_n[\min\{\theta^*_{n-1,n}, s\} - \min\{\theta^*_{n,n+1}, s\}] \tag{2}$$

The price of the asset is determined by the expectations of all agents. Even though only the most optimistic agent purchases the asset directly, the expectations of the other agents are important in determining the price because expectations determine each agent's willingness to lend. More lending means the most optimistic agent has more funds to use in purchasing the asset, so the market clearing price must be higher.

In particular, since

$$E_n[\min\{\theta, s\} - \min\{\theta', s\}] = (1 - F_n(\theta^*_{n-1,n}))\theta^*_{n-1,n} + \int_{\theta^*_{n,n+1}}^{\theta^*_{n-1,n}} s f_n(s) ds$$

the asset's price is determined by two features of agent n's beliefs: the upper tail probability $1 - F_n(\theta^*_{n-1,n})$ and the perceived risk $\int_{\theta^*_{n,n+1}}^{\theta^*_{n-1,n}} s f_n(s) ds$ (which is also determined by the lower tail probability). When deciding how much to lend, it makes no difference to type n how payoffs are distributed above $\theta^*_{n-1,n}$. All that matters is the perceived probability that the loan will be repaid in full. Below $\theta^*_{n-1,n}$, the distribution of payoffs does affect how much type n chooses to lend, as type n bears payoff risk in that region. These properties will be crucial in the analysis of comparative statics in Section 4.

The values r_n, which represent perceived return on wealth for each type, merit some explanation as well. Recall that $\frac{r_n}{r_{n+1}} = \frac{1-F_n(\theta^*_{n,n+1})}{1-F_{n+1}(\theta^*_{n,n+1})}$. This formula can be interpreted as follows: at the margin, type n agents decide whether to promise an extra unit of repayment θ at $t = 1$ to type $n + 1$ agents. A small change in loan riskiness represents one extra dollar of repayment whenever $s \geq \theta$, since the agent will walk away from the loan regardless whenever $s < \theta$. The amount type $n+1$ is willing to lend at $t = 0$ for that repayment is $\frac{1-F_{n+1}(\theta)}{r_{n+1}}$, and the cost of that repayment to type n is $\frac{1-F_n(\theta)}{r_n}$. Therefore type n's repayment choice must satisfy $\frac{r_n}{r_{n+1}} = \frac{1-F_n(\theta^*_{n,n+1})}{1-F_{n+1}(\theta^*_{n,n+1})}$.

Observe that $\frac{1}{r_n} = \prod_{k=n}^{N-1} \frac{r_{k+1}}{r_k} = \prod_{k=n}^{N-1} \frac{1-F_{k+1}(\theta^*_{k,k+1})}{1-F_k(\theta^*_{k,k+1})}$. The beliefs of type N then appear to be important in determining the return on wealth for all agents, since r_n has a factor of $1-F_N(\theta^*_{N-1,N})$ for all n. This is a leverage effect: the returns on wealth of type n agents are determined in part by their ability to borrow, which is in turn related to type $n+1$'s ability to borrow, and so forth all the way down to type N. If type N is sufficiently pessimistic, the pyramid of lending can effectively collapse, leaving type n agents to fund asset purchases using little more than their endowments. Indeed, type n's leverage can be written as

$$L_n = 1 + \frac{q(\theta^*_{n,n+1})}{w_j} = 1 + \frac{\sum_{m>n} \frac{1}{r_m} E_m[\min\{\theta^*_{m-1,m}, s\} - \min\{\theta^*_{m,m+1}, s\}]}{w_j} \tag{3}$$

When agents of type N are very pessimistic, the term $1 - F_N(\theta^*_{N-1,N})$, which appears in all terms of the sum in the numerator, is small. Leverage should thus be lower when agents of type N become more pessimistic.

Beliefs are also an important determinant of the marginal interest rates perceived by agents (that is, the interest rate on each extra dollar of $t = 0$ borrowing). In equilibrium, the marginal perceived interest rate R_n must equal the perceived return on a dollar r_n for each type n. The marginal interest rate can therefore be written as

$$R_n = \frac{1}{1 - F_N(\theta^*_{N-1,N})} \left(\prod_{m=n+1}^{N-1} \frac{1 - F_m(\theta^*_{m,m+1})}{1 - F_m(\theta^*_{m-1,m})} \right) (1 - F_n(\theta^*_{n,n+1})) \tag{4}$$

The first two terms are the inverse probabilities that types $m > n$ in the lending chain assign to being repaid conditional on not defaulting themselves. The third term is the inverse of type n's subjective probability of default. When pessimistic types think they are not likely to be repaid, they charge high interest rates on loans. On the other hand, when type n agents believe they will likely default (i.e., $F_n(\theta^*_{n,n+1})$ is high), they perceive a low interest rate because they believe that there is a good chance they will never repay the loan.

Of course, all of the preceding analysis relies on partial equilibrium logic, since the values $\theta^*_{n,n+1}$ are endogenous objects. However, these partial equilibrium effects provide important intuition for the results of Section 4, and for the most part, these intuitions carry through in the full general equilibrium of the economy.

3.4 Equilibrium Characterization

In this subsection, I briefly list and explain the equations defining an equilibrium. The proof of Proposition 3.3 shows that there exists some K such that the following equations hold for $n \in \{1, \ldots, K-1\}$:

$$w_n = \left(\prod_{k=n}^{N-1} \frac{1 - F_{k+1}(\theta^*_{k,k+1})}{1 - F_k(\theta^*_{k,k+1})} \right) E_n[\min\{\theta^*_{n-1,n}, s\} - \min\{\theta^*_{n,n+1}, s\}] \tag{5}$$

where $\theta^*_{0,1} \equiv \bar{s}$. The values $r_n = \prod_{k=n}^{N-1} \frac{1-F_k(\theta^*_{k,k+1})}{1-F_{k+1}(\theta^*_{k,k+1})}$ must be greater than or equal to 1 for $n < K-1$ and equal to 1 for all $n \geq K$ if $K < N$. Any vector $(\theta^*_{1,2}, \ldots, \theta^*_{K-1,K})$ satisfying this system of equations yields an equilibrium. This equation says that each type's wealth is equal to its expected payoff in equilibrium divided by its expected return on wealth. Type n's expected return on wealth, as mentioned in Section 3.3, is a function of each higher type's subjective probability of default.

Observe that equation (**??**) can be rewritten as

$$w_n = (1-F_N(\theta^*_{N-1,N}))\left(\prod_{m=n+1}^{N-1} \frac{1-F_m(\theta^*_{m-1,m})}{1-F_m(\theta^*_{m,m+1})}\right) E_n[\min\{\theta^*_{n-1,n}, s\} - \theta^*_{n,n+1} | s \geq \theta^*_{n,n+1}] \quad (6)$$

Equation (**??**) can therefore be rewritten in terms that only depend on each type's beliefs conditional on not defaulting. This observation is interesting in light of other observations in the literature, such as the claim in Simsek (2013) that it is the *skewness* of beliefs that determines asset prices. This formula shows that it is really each type's belief about the upside of its portfolio that pins down the asset price.

4 What Does Intermediation Do?

This model raises an obvious question of both theoretical and practical interest: how does intermediation affect equilibrium allocations? The main result of this section will show that the increase in the capitalization of an intermediary (or, similarly, a lengthening of the intermediation chain) tends to drive up the riskiness of loans taken by more optimistic agents and lower the riskiness of the loans made by more pessimistic agents. In Section 5's discussion of applications this characterization will play a key role.

In order to prove the results discussed above, the following definition will be helpful:

Definition 4.1. *Aggregate wealth is abundant if* $\sum_{i=1}^{N} w_i > E_1[s]$. *The wealth of a subset of types* $S \subset \{1, \ldots, N\}$ *is scarce if* $\sum_{i \in S} w_i < E_N[s]$.

Intuitively, when aggregate wealth is abundant, some type must hold cash because the price of the asset can never exceed the expectation of the most optimistic agent. As a consequence, some type is indifferent between lending and holding cash, meaning that this type always breaks even in expectation when lending. When the wealth of a subset of types is scarce, that group of types must borrow from wealthier agents if they are net purchasers of the asset, since the price of the asset can never be lower than the expectation of the least optimistic agent when aggregate wealth is abundant.

Suppose there are N types with priors $F_1(s), \ldots, F_N(s)$ satisfying Assumption 1. If K of these types have initial wealth equal to 0, the maximum possible length of a lending chain is $N - K$. The length of the lending chain can increase if the wealth of some type with zero wealth is increased. Proposition 4.2 is motivated by this notion of chain length.

Proposition 4.2. *When total wealth is abundant,*

1. *The face value of loans made to type n agents is increasing in the wealth of types $m > n$;*

2. *The riskiness of and the perceived marginal interest rate on the debt contract purchased by type n agents is decreasing in the wealth of types $m < n$.*

The logic behind this proposition is straightforward. When type n intermediaries have more wealth, they bid up the amount they are willing to lend to type $n - 1$ agents, who in turn lend more to type $n-2$, and so forth. The increase in type n's demand for debt contracts propagates all the way up the chain and allows more optimistic types to take riskier loans at lower interest rates. Intermediaries also provide a cash buffer for the most pessimistic lenders, meaning that those agents are willing to lend at lower marginal interest rates. When type n agents are wealthier, they are able to bear more of the risk associated with holding the collateral. Then type $n + 1$ agents make safer loans to type n agents, thereby reducing their borrowing needs and allowing them to take safer loans from type $n + 2$ agents. As intermediary capitalization increases, the riskiness of the end-lenders' loans approaches zero. Intermediation effectively tranches the risky asset's payoffs to such an extent that it produces almost perfectly safe assets for pessimists.

An increase in the length of the intermediation chain can be interpreted as follows: if some intermediate type starts with wealth equal to zero, increasing that type's wealth to some positive value will cause a discrete change in the length of the chain. If the length of the chain is increased in this way, then, the leverage of borrowers will increase and the interest rate charged by lenders will decrease. The following corollary follows naturally from the fact that increasing the length of the lending chain is equivalent to increasing the wealth of some intermediary in this setting.

Corollary 4.3. *The leverage of the most optimistic type is increasing in the number of intermediaries. The marginal interest rate charged by the most pessimistic lender and paid by the most optimistic borrower is decreasing in the number of intermediaries.*

Financial intermediation chains and the reuse of collateral are key to the simultaneous increases in leverage and the supply of safe assets. Indeed, in this model intermediary capitalization, leverage, and the safety of end-lenders' portfolios all move together. This relationship is in accordance with some recent stylized facts. Lenel (2017) documents that the supply of safe assets produced by the private sector underwent an unprecedented expansion prior to the financial crisis of 2007-2009. Geanakoplos (2009), on the other hand, shows that margins on the contracts backing those safe assets were countercyclical. Tranched securities are the archetypical example of this phenomenon: if the riskiness of the underlying assets is held fixed, a greater number of tranches must mean that the lower tranches are safer and the upper tranches are riskier.

Proposition 4.2 highlights a fundamental property of intermediation in heterogeneous belief models. In any such model, intermediation will focus underlying risks on agents who are most willing to bear them. When optimists are poor and there are no intermediaries, pessimists are forced to bear some risk because optimists can only afford to purchase the asset if they take risky loans. When intermediaries enter, they can act as optimists' agents by providing them with funding while protecting more pessimistic agents from default. Intermediation focuses borrowers' risks into the region of the state space where they are most optimistic relative to other agents.

It is also useful to characterize intermediation in terms of agents' perceived returns on wealth r_i, which are endogenous quantities. The following result is similar to Proposition 4.2 in spirit and will provide insight during the discussion of applications in Section 5:

Proposition 4.4. *Fix some type $i \in \{2, \ldots, N-1\}$. Holding the returns on wealth of types $j \neq i$ constant, a decrease in r_i raises the riskiness of loans taken by type $i-1$ agents and decreases the riskiness of loans made by type $i+1$ agents.*

Recall that there is a mapping from returns on wealth to the distribution of wealth– in particular, the returns on wealth of a given type are inversely related to that type's wealth. This proposition is then akin to a local rephrasing of Proposition 4.2, since it relates intermediation to the positions taken by a given type rather than those taken by all types.

5 Applications of the model

In this section I present several applications of the model. I show that the model generates contagion in margin spikes both along the chain of lending and across asset classes, that leverage cycles coincide with cycles of safe asset production by intermediaries, and analyze the consequences of financial innovations by comparing the model to a benchmark in which it is not permitted to use debt as collateral.

5.1 Contagion in Margins

While contagion is often discussed in the context of asset prices, during the financial crisis of 2007-2009 spikes in margins for mortgage-backed CDOs propagated to other markets as well. As Gorton and Metrick (2012) document, in bilateral repo markets margins rose in essentially all risky asset classes. Furthermore, the increase in CDO margins coincided with a decrease in loan-to-value ratios in the housing market, which is a phenomenon that Geanakoplos (2010) terms the *double leverage cycle*: increases in the capital requirements for purchasing asset-backed securities translate into increases in margins on the securities themselves, which leads to instability both in the market where intermediaries lend and in the market in which they borrow. I show that my model explains both of these phenomena in a straightforward way.

There are two forces that determine the margin put down by agents of type i in this model: the beliefs of other agents and the wealth of other wealth-constrained agents. Broadly speaking, the margin paid by a given type of agent rises when either other agents become more pessimistic or other agents are more wealth-constrained. For now, I focus on a shock to the pessimism of the end-lenders who hold the safe tranche (the most pessimistic type) as the driver of margins, since this type of shock is the one most often discussed in the literature on asset pricing with endogenous collateral constraints.

I first show that a shock that "scares" pessimists into becoming even more pessimistic leads to all types $i = 1, \ldots, N-1$ paying higher margins on the securities they purchase. Proposition 5.1 provides a formal statement:

Proposition 5.1. *Suppose that in the equilibrium with initial endowments $w_1, \ldots, w_N$ and beliefs $F_1, \ldots, F_N$, the margin paid by type i agents is*

$$m_i = \frac{\frac{1}{r_i} E_i[\min\{\theta^*_{i-1}, s\} - \min\{\theta^*_i, s\}]}{\sum\limits_{j=i}^{N} \frac{1}{r_j} E_j[\min\{\theta^*_{j-1}, s\} - \min\{\theta^*_j, s\}]}$$

. Then if the belief of type N is changed to $\tilde{F}_N$ such that $\tilde{F}_N(s) \geq F_N(s)$ for all s and all other beliefs and endowments are held fixed, the new equilibrium margins $\{\tilde{m}_i\}_{i=1}^{N-1}$ satisfy $\tilde{m}_i \geq m_i$ for all i.

This model therefore features not only double leverage cycles in which the margins on two different securities are linked, but rather leverage cycles of degree N where the margins on securities at all levels of the lending chain move together. Empirically, this corresponds to comovement in margins on all asset-backed debt of all different ratings, which did occur during the collapse of the securitized products market in 2008.

Of course, increased pessimism by those holding the safest tranche of debt will lead pessimists to lend less against any collateral. This feature is shared by essentially all models with endogenous collateral constraints. The feature that is particular to a model with endogenous *chains* of lending is that once the lenders at the end of the chain become less willing to lend, intermediaries increase their margin requirements as well. The key insight behind why an increase in margins at the bottom level of the chain leads to a spike in margins all along the chain is that intermediary lenders are wealth-constrained. When cash-rich pessimists suddenly stop lending against the debt contracts purchased by these intermediaries, they are simply unable to lend as much to end-borrowers (optimists) because their leverage has decreased. Put differently, the spike in margins at the bottom of the chain effectively increases the cost of capital for intermediaries because pessimists demand higher interest payments, so intermediaries are less willing to pay optimists for a given promised future cash flow, which decreases margins at the top of the chain as well.

This double leverage cycle result is directly linked to the results of Section 4 on the consequences of intermediation. Intermediaries must always hold a cash buffer to protect pessimists from negative shocks to the asset's payoff. When pessimists feel sufficiently protected, they lend large quantities to intermediaries, which in turn enables them to turn around and lend to optimists, which generates high leverage at the top of the chain. However, when pessimists perceive additional tail risk on the asset, they require a larger cash buffer to make a loan of a given size. Barring a positive shock to intermediary capitalization, intermediaries are unable to provide a larger cash buffer than previously, so the size of loans made by pessimists to intermediaries (and thus the size of loans made by intermediaries to optimists) must decrease.

I now show that in an extension of the model, a shock to pessimism of the type in Proposition 5.1 can cause an increase in margins across asset classes even when payoffs are uncorrelated across classes. Suppose that there are two assets A and B (each in unit supply) that yield payoffs s_A, s_B at $t = 1$. There are four types of investors: type 1_A investors, type 1_B investors, type 2 investors, and type 3 investors. Type 1_A investors have the most optimistic assessment $F_1^A(s_A)$ about asset A's payoffs, type 2 and type 1_B investors share the second-most optimistic beliefs $F_2^A(s_A)$, and type 3 investors are the most pessimistic about A's payoff, with beliefs $F_3^A(s_A)$. In market B, agents' beliefs are analogous to those in market A (with the roles of 1_A and 1_B flipped). The contracts are as in the baseline model. A debt contract may be collateralized by either a unit of asset A, a unit of asset B, or another debt contract. Hence in market A, 1_A will borrow from 2 who borrows from 3, whereas in market B, 1_B borrows from 2 who borrows from 3. Both 2 and 3 are "crossover investors" in the sense that they invest in both markets, but the critical feature of this model will be that wealth-constrained type 2 agents lend to optimists and determine margins in both markets.

Type 1_A and 1_B investors are relatively more optimistic about one of the two assets, so they put their entire endowment into their respective markets.

The main result in this extension of the baseline model is that a pessimistic shock to type 3's belief about asset A causes margins paid by both type 1_A agents and type 1_B agents to rise. This result is summarized in Proposition 5.2.

Proposition 5.2. *Suppose that in the equilibrium with initial endowments w_{1A}, w_{1B}, w_2, w_3 and beliefs $\{F_1^k, F_2^k, F_3^k\}$ for $k \in \{A, B\}$, margins paid by end-borrowers are m_A and m_B. Then in an alternative equilibrium where only 3's beliefs about asset A change to $\tilde{F}_3^A$ such that $\tilde{F}_3^A(s) \geq F_3^A(s)$ for all s, margins increase in both markets; i.e., $\tilde{m}_A \geq m_A$ and $\tilde{m}_B \geq m_B$.*

Proposition 5.2 shows that a disruption to the collateral capacity of asset A also reduces the collateral capacity of asset B regardless of how the payoffs of asset B are related to those of asset A. Although contagion in asset prices across markets is a common feature of models with wealth-constrained investors who trade multiple asset classes, contagion in margins is not. For example, in a setting of Simsek (2013) in which there is no scope for intermediation, an shock to pessimism about asset A would cause end-borrowers to move their wealth out of asset B and into asset A to cushion the crash in prices, which would decrease margins in market B rather than increasing them. In a model with information frictions like that in Dang, Gorton, and Hölmstrom (2013), a shock increases margins across several asset classes only if that shock affects the value of information about a fundamental common to the payoffs in all of those asset classes. Finally, in Brunnermeier and Pedersen (2009), margins increase across asset classes only because lenders misinterpret the wealth effect that leads to departures of prices from fundamentals across markets as a shock to the fundamentals of assets in all those markets.

By comparison, this explanation seems much more parsimonious than those offered in the literature, as it achieves contagion in margins without any assumption about the correlation of assets' payoffs or the bounded rationality of lenders. The mechanism of Proposition 5.2 relies on the simple fact that *lenders* may be wealth constrained, and when those lenders participate in several markets, they will be able to lend less against collateral in all markets. This explanation certainly seems plausible in the case of the bilateral repo market during the "run on repo" of 2008 studied by Gorton and Metrick (2012). In that market, leveraged investors such as banks and hedge funds obtain funding from each other rather than from institutions with safer portfolios such as MMMFs and securities lending firms. In the context of this model, the observed rise in margins on all risky securities during the run on repo is consistent with financial stress in the intermediary sector.

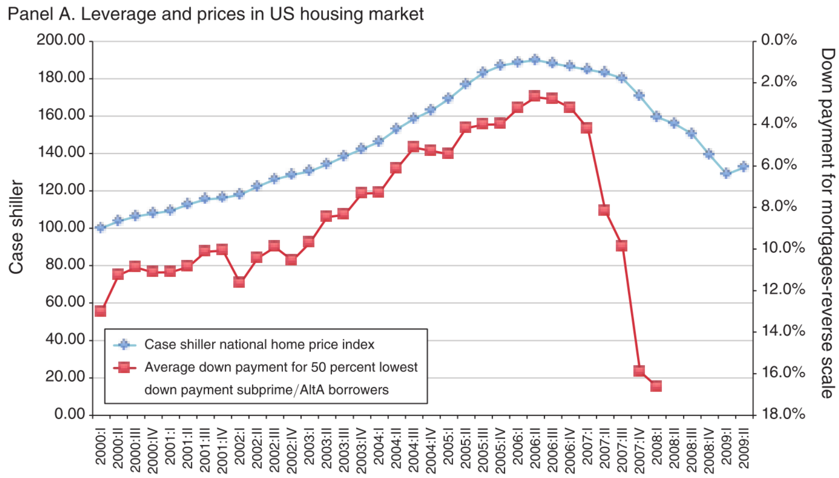

Figure 2: House prices (in blue) and loan-to-value ratios (in red) during the period 2000-2009. Loan-to-value ratios are plotted as margins on an inverse scale.

5.2 Leverage and Safe Asset Production Cycles

During the boom in securitization leading up to 2007, both leverage and private-label safe asset production exploded along with real estate prices and investment. After the collapse of the securitized products market, private production of safe assets and loan-to-value ratios collapsed. Figures 2 and 3, taken from Fostel and Geanakoplos (2016) and Lenel (2017), respectively, depict these patterns. I now show that the baseline model augmented to allow for investment explains this episode as a consequence of the rise and subsequent fall of net worth in the intermediary sector, which in this case can be thought of as the shadow banking system.

In order to isolate the effect of intermediary net worth on leverage and safe asset production, I modify the model by assuming that there are only three types ordered by their optimism and that type 1 agents and type 3 agents have large endowments. However, type 1 agents are relatively more impatient than type 3 agents: their utility is $u_1(c_0, c_1) = r_1 c_0 + c_1$ with $r_1 > 1$, whereas type 2 and 3 agents have utility $u_2(c_0, c_1) = u_3(c_0, c_1) = c_0 + c_1$. The assumption that the initial endowments of types 1 and 3 are large implies that their discount factors are essentially exogenous: type 1 will price any asset with payoff $\phi(s)$ as $\frac{1}{r_1}E_1[\phi(s)]$, and type 3 agents price assets as $E_3[\phi(s)]$ as in the baseline model. The assumption that type 1 is impatient can alternatively be seen as an assumption that type 1 agents have a superior outside investment option that yields expected returns r_1. Ultimately, this assumption just serves to capture the idea that type 1 will invest in the asset only if it is possible to lever up

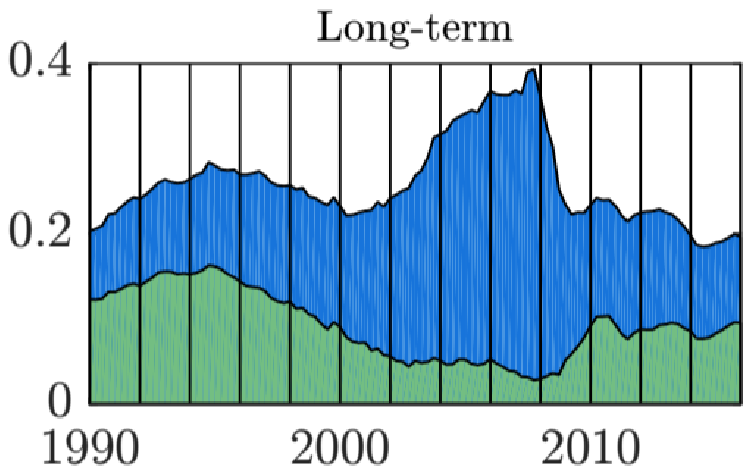

Figure 3: Private-label long-maturity safe asset production (in blue) and long-maturity Treasury debt (in green) during the period 1990-2017.

and obtain high expected returns.

I also introduce investment into the baseline model to capture the boom and bust in investment that coincided with the rise and fall of leverage and private safe asset production. Formally, there is an investment technology that allows any agent to produce a unit of the asset by paying a price $p(q)$, where q is the quantity of the asset that has been produced in aggregate and p is a continuous increasing function. This assumption does not change any of the conclusions of the baseline model– it is still the case that type 1 will borrow from type 2, who uses that debt contract as collateral to borrow from type 3. The only differences between this model and the baseline model are that (1) the returns on wealth perceived by type 1 are exogenous, and (2) the supply of the asset is endogenous.

The equations that pin down equilibrium in this extension of the model are

$$r_1 = \frac{1 - F_1(\theta_1^*)}{1 - F_2(\theta_1^*)} \frac{1 - F_2(\theta_2^*)}{1 - F_3(\theta_2^*)}$$

$$w_2 = \frac{1 - F_3(\theta_2^*)}{1 - F_2(\theta_2^*)} q(p) E_2[\min\{\theta_1^*, s\} - \min\{\theta_2^*, s\}]$$

$$p = \frac{1}{r_1} E_1[\max\{s - \theta_1^*, 0\}] + \frac{1 - F_3(\theta_2^*)}{1 - F_2(\theta_2^*)} q(p) E_2[\min\{\theta_1^*, s\} - \min\{\theta_2^*, s\}] + E_3[\min\{\theta_2^*, s\}]$$

where $q(p)$ is the inverse of $p(q)$. It can easily be shown that these equations yield a unique solution for $(\theta_1^*, \theta_2^*, p)$.

The main result regarding the model with investment links intermediary wealth v_2, leverage, safe asset creation, investment, and asset prices:

Proposition 5.3. *The margins m_1 paid by type 1 agents on a unit of the asset, the riskiness θ_1^* of loans taken by type 1 agents, investment q, and prices p are all increasing in intermediary net worth w_2. The riskiness of loans made by type 3 agents θ_2^* is decreasing in w_2.*

The logic underlying this result is similar to the intuition behind Proposition 4.2. Intermediation by agents who are willing to bear risk boosts the collateral capacity of the asset, which raises its price, and allows those least willing to bear risk to make safer loans. When investment is introduced, the boom in asset prices additionally leads to increased investment, since it becomes more profitable to invest than to purchase the asset directly.

The other main benefit of this alteration of the model, however, is that it is numerically solvable in a dynamic setting. I present an example that suggests the static result of Proposition 5.3 goes through to a dynamic setting even when intermediaries might be incentivized to "keep their powder dry" by making safer loans and not exposing themselves to risk early on. Intermediary net worth is correlated the shocks to the asset's payoff, which yields a dynamic version of the proposition's result.

Example 5.4: There are three periods, $t = 0, 1, 2$. There are three types of agents with the preferences specified above. In each period $t \in \{1, 2\}$, a unit of the asset pays off $s \in \{1, 2, 3, 4, 5\}$ and depreciates completely. The asset's payoffs (as perceived by each type) are iid across periods. Agents' beliefs satisfy the hazard rate ordering.[1] Type 2 agents start at $t = 0$ with an endowment of $w_2 = 20$ dollars. The function that pins down the cost of investment is $q(p) = 2 + 4p$.

As shown in Figure 4, the results in the dynamic model are exactly those suggested by Proposition 5.3. When the state realized at $t = 1$ is good, the intermediary is wealthy at $t = 1$, investment booms, the asset price rises, optimists are able to take riskier loans, and pessimists hold only riskless debt. By contrast, when the realized asset payoff at $t = 1$ is low, the intermediary is wealth constrained and leverage, prices, and investment collapse while pessimists are forced to hold risky debt. When the worst outcome is realized, in fact, the intermediary's net worth is completely wiped out and pessimists have to lend directly to optimists. These results broadly capture the trends in the run-up to and fallout of the crash displayed in Figures 2 and 3.

[1] I do not explicitly write down agents' beliefs in order to conserve space. It makes no difference that the agents believe the payoffs of the asset take values in a discrete set– all of the conclusions of the baseline model go through.

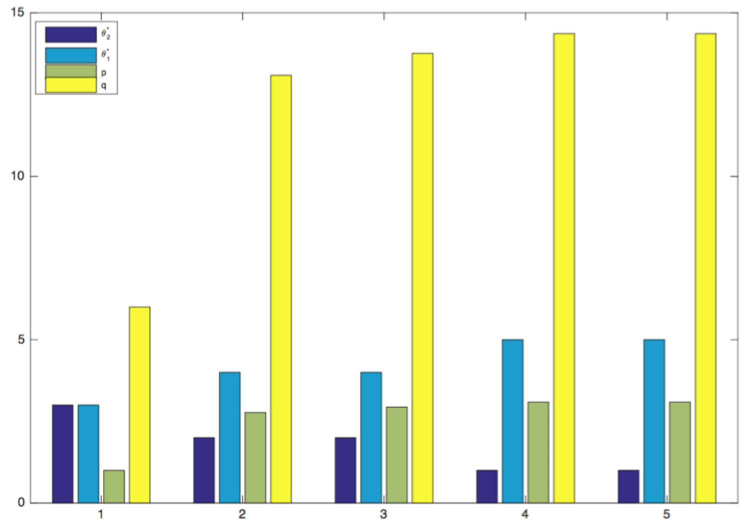

Figure 4: Riskiness of loans taken by type 2 θ_2^*, riskiness of loans made by type 2 θ_1^*, asset prices p, and investment quantities q in Example 5.4 as a function of the realized state $s \in \{1, 2, 3, 4, 5\}$ at $t = 1$.

5.3 The Impact of Financial Innovation

In this section, I introduce an alternative economy in which it is not possible to use debt contracts as collateral and ask how the benchmark economy differs from this alternative one. I primarily compare risk-taking in the two economies and show that some agent always takes greater risk in the benchmark economy than in the alternative economy.

The model is identical to that described in Section 2.1 with the exception of the space of assets that are traded. In this version of the model, the only admissible collateral is the physical asset a. Therefore, there are two types of positions that agents may hold: they may borrow using the asset as collateral and promise to repay θ in the next period, or they may lend to a borrower and take possession of the asset in case of default. In other words, when a unit of the asset is purchased, the borrower receives a payoff $\max\{s - \theta, 0\}$ and the lender receives $\min\{\theta, s\}$ when the loan has riskiness θ.

The optimization problem of a type-n agent is

$$\max_{a,\mu_+,\mu_-,c} aE_n[s] + c + \int E_n[\min\{\theta, s\}]d\mu_+(\theta) - \int E_n[\min\{\theta, s\}]d\mu_-(\theta)$$
$$s.t.\ w_n = pa + c + \int q(\theta)d\mu_+(\theta) - \int q(\theta)d\mu_-(\theta)$$
$$a \geq \int d\mu_-(\theta),\ \mu_+(\theta) \geq 0,\ \mu_-(\theta) \geq 0$$

This differs from the optimization problem in Section 2.2 only in that debt contracts cannot be used as collateral: in this case, the total number of borrowing contracts must be less than or equal to the number of units of the asset a purchased by the agent.

An alternative formulation of this problem is

$$\max_{\mu_b,\mu_l,c} \int E_n[\max\{s-\theta,0\}]d\mu_b(\theta) + \int E_n[\min\{\theta,s\}]d\mu_l(\theta) + c$$
$$s.t. w_n = \int (p-q(\theta))d\mu_b(\theta) + \int q(\theta)d\mu_l(\theta) + c,\ \mu_b(\theta) \geq 0, \mu_l(\theta) \geq 0$$

In this formulation, $\mu_b(\theta)$ represents the number of borrowing contracts of riskiness θ collateralized by a unit of the asset and $\mu_l(\theta)$ represents the number of lending contracts of riskiness θ held by the agent.

An equilibrium is defined in the obvious way. From now on, for ease of exposition I assume that there are only three belief types, although the model can easily be extended to include more. In this setting, as long as types 1 and 2 are wealth-constrained there are only two possible types of equilibria: ones in which both types 1 and 2 purchase the asset and borrow from type 3, and ones in which only type 1 buys the asset and borrows from types 2 and 3. Proposition 5.5 formalizes the main result.

Proposition 5.5. *If, given fixed initial endowments and beliefs, type 2 agents buy the asset directly in the equilibrium of the alternative economy, type 1 agents take riskier loans in the benchmark economy than in the alternative economy. If type 2 agents lend to type 1 agents in the equilibrium of the alternative economy, type 2 agents take greater risk in the benchmark economy than in the alternative economy. The asset price is always higher in the benchmark economy*

This result, while almost obvious, will have important implications for the dynamics of asset prices. When natural intermediaries are lenders in the setting without debt contracts as collateral, they take advantage of financial innovation by levering up to lend to optimists. On the other hand, when these natural intermediaries are buyers of the asset before financial innovation, the introduction of debt contracts as collateral actually makes them take a slightly safer position: they borrow to make risky loans to optimists rather than buying the asset directly, which in turn puts optimists in a much riskier position.

As in the previous subsection, I construct a dynamic numerical example to illustrate this effect.

Example 5.6: There are three periods, $t = 0, 1, 2$. Agents with a desire to consume immediately sell a single unit of an asset in periods 0 and 1. Agents of types 1, 2, and 3 buy this asset and trade collateralized debt contracts in the first two periods. Type 1 and 3 agents are short-lived, meaning a new generation of type 1 and 3 agents enters the market at time 1, but type 2 agents are long-lived. This assumption focuses the example on the consequences of risk-taking by intermediaries rather than end-borrowers. The asset's payoff in each period falls in the set $s \in \{u, m, d\}$, where $u = 1$, $m = 0.5$, and $d = 0.2$. At $t = 0$, agents have the following beliefs:

$$Pr_1(s_0 = u) = 0.8,\ Pr_1(s_0 = m) = 0.1,\ Pr_1(s_0 = d) = 0.1$$

$$Pr_2(s_0 = u) = 0.1,\ Pr_2(s_0 = m) = 0.6,\ Pr_2(s_0 = d) = 0.3$$

$$Pr_3(s_0 = u) = 0.02,\ Pr_2(s_0 = m) = 0.18,\ Pr_2(s_0 = d) = 0.8$$

If the realization of s_0 is either u or d, all agents agree that at $t = 1$ the asset will pay u. On the other hand, if $s_0 = d$, agents' beliefs about s_1 are identical to their $t = 0$ beliefs about s_0. Type 1 agents are endowed with 0.16 dollars in each period, and type 2 agents start with an endowment of 0.2 dollars at $t = 0$.

At $t = 1$, if the realization of s_0 was u or m, the asset's price will be equal to 1. In order to solve the example, it is necessary to find the allocation and prices at $t = 1$ after d is realized and then work backwards to determine the allocation and prices at $t = 0$. When debt cannot be used as collateral, in this example types 2 and 3 lend to type 1. At $t = 1$ after d is realized, type 2 still has some remaining wealth in the alternative economy without debt as collateral but is completely wiped out in the benchmark model. The asset price at $t = 0$ is 0.56 in the benchmark economy, whereas in the economy without debt as collateral, the asset price at $t = 0$ is 0.42. In the initial period, the asset's price is higher in the benchmark economy because intermediaries sell off the safe tranche of debt that they hold to pessimists and then use that additional capital to lever up and lend to optimists, at which point they sell off the lower tranche again, and so forth. This cycle of lending and levering up greatly increases the asset's collateral capacity. However, once d is realized, the asset's price crashes to 0.36 in the benchmark economy, but in the alternative economy it falls to only 0.40. The price in the benchmark economy is both higher in the initial period and lower in the second period, and the volatility exceeds that in the alternative economy by an order of magnitude. Volatility is higher in the alternative economy simply because allowing lenders to lever up may mean that they go bankrupt and are absent when the market crashes, which incidentally is when they are most needed to prop up asset prices. This example thus illustrates how the ability of the financial sector to use leverage and stretch collateral among several parties may lead to large run-ups and precipitous falls in asset prices.

6 Discussion

In this section I briefly address issues regarding the plausibility of the modeling choices and how they affect the results. I consider two issues: the introduction of alternative sets of securities and the use of differences in beliefs as a motive for trade.

6.1 Alternative Asset Markets

It is not immediately clear from the discussion in previous sections how the results would change if the set of feasible portfolios did not consist solely of leveraged positions in collateralized debt contracts. For example, if agents were permitted to trade Arrow-Debreu securities, would chains of intermediation emerge? Would information aggregation be impaired in situations with long chains of intermediation?

The results presented above are quite general. The fundamental ingredients of the model are the following: a function $g_n(\theta) = E_n[\min\{\theta, s\}]$ of a parameter θ corresponding to each agent's expected asset payoffs and an assumption on the ordering of beliefs, which in this case is that

$$-\frac{d}{d\theta}\log(g_n'(\theta)) = -\frac{g_n''(\theta)}{g_n'(\theta)} = \frac{f_n(\theta)}{1 - F_n(\theta)}$$

is ordered by n. As long as agents are permitted to take matched long-short positions in contracts with payoffs $g_n(\theta)$ (as in this model) they can take positions with payoffs $g_n'(\theta)$ (which in this model is $1 - F_n(\theta)$). Any feasible payoff can then be written as an integral $\int\limits_{\theta\in\Theta} g_n'(\theta)d\theta$ for some set Θ. When agents have perceived returns on wealth $\{r_n\}_{n=1}^N$, type n will price positions such that $\frac{1}{r_n}g_n'(\theta) = \max\limits_m \frac{1}{r_m}g_m'(\theta)$. Note that in this discussion limited commitment is not explicitly mentioned– it is only necessary in order to motivate the specific form of the payoff function.

A simple way to construct such a model with complete markets is to take $g(\theta) = E_n[\mathbf{1}\{s \leq \theta\}]$. Then $g_n'(\theta) = f_n(\theta)$, so agents are able to synthesize Arrow-Debreu securities. In this context, the necessary assumption on beliefs is that $\frac{g_n''(\theta)}{g_n'(\theta)} = \frac{f_n'(\theta)}{f_n(\theta)}$ is a decreasing function of n, which is equivalent to assuming the monotone likelihood ratio property. Even in an environment with complete markets, then, the ordering of beliefs has a natural interpretation. When this restriction on beliefs is imposed, agents form a chain: the most pessimistic agent holds claims on the worst states, the next-most pessimistic agent holds claims on the next interval of states, and so forth.

I use collateralized debt contracts in my analysis because they are the predominant financial instrument used to fund investments in several markets, such as the repo market and the housing market. This set of instruments thus allows for the analysis of empirically relevant issues. On the other hand, it is unclear to what an Arrow-Debreu contract corresponds in the data. There are also other reasons outside the scope of this model that one might want to restrict the analysis to simple collateralized debt contracts. For example, Dang, Gorton, and Holmstrm (2013) show that debt contracts optimally minimize the loss of liquidity due to asymmetric information.

6.2 Beliefs as a Motive for Trade

In the literature, heterogeneous beliefs often serve in as an ad hoc method of introducing a motive for borrowing and lending. However, the assumptions on beliefs in this model may raise some concerns. First, they are somewhat stronger than those in a binomial model, for example, since a binomial model requires only that agents disagree about one state, whereas here a hazard rate ordering must be satisfied over the entire state space. Second, it may not seem natural to equate financial intermediaries with agents who have intermediate beliefs. I discuss two alternative modeling choices – risk limits and ambiguity aversion – that take a step towards addressing these concerns yet result in an isomorphic model with exactly the same predictions.

Suppose all agents have a common prior $F(s)$ but are subject to risk limits. An extreme example of risk limits is a case in which agents of type n are not allowed to take any position that takes a loss with probability greater than $F(\hat{\theta}_n)$. In this case, agents act as if they had beliefs satisfying $1 - F_n(s) = \lambda_n(s)(1 - F(s))$ where

$$\lambda_n(s) = \begin{cases} 1 & s \leq \hat{\theta}_n \\ 0 & s > \hat{\theta}_n \end{cases}$$

That is, they act as if they believe states higher than $\hat{\theta}_n$ are impossible. More generally, one could assume that type n agents are subject to compensation schemes and restrictions that incentivizes them to act as if they had beliefs satisfying $1 - F_n(s) = \lambda_n(s)(1 - F(s))$. In this case, the hazard rate ordering reduces to the assumption that $\frac{\lambda_n'(s)}{\lambda_n(s)}$ is a decreasing function of n.

Such risk limits are not uncommon in the financial industry, and it is natural to think of lenders such as money market funds as having tight risk limits and borrowers such as hedge funds as having loose limits. In practice, intermediaries such as investment banks are able to take greater risks than money market funds and can therefore serve as a link between hedge funds and sources of funding. In a model with this type of constraint, intermediation would be essential in facilitating trade between cash-constrained borrowers and lenders who are unable to bear the risk associated with the volatility of the underlying asset's payoffs.

The interpretation of borrowing constraints would be different, but the implications would be exactly the same as those of the baseline model.

One could also assume that investors are ambiguity averse and evaluate payoffs according to their worst-case belief. If some investors face significant model uncertainty whereas others have a better grasp of the statistical properties of the asset's payoffs, it is natural to assume that each type n considers a set of possible priors μ_n such that $\mu_1 \subset \mu_2 \subset \cdots \subset \mu_N$. Under this assumption, type n is more certain of the distribution than type $n+1$ for all n. The worst-case distribution of type n is therefore always more optimistic than the worst-case distribution of type $n+1$. If the distributions within each μ_n can be ordered by their hazard rates, then once again this model is isomorphic to the baseline model. The assumption that people are ambiguity averse has experimental support, and it is not unreasonable to assume that natural lenders like money market funds would be uncertain about the payoffs of risky assets relative to hedge funds that employ traders who specialize in trading those assets.

Thus there are several equivalent ways of arriving at the same model. The assumption of heterogeneity of beliefs serves to simplify the exposition, but it really amounts to assuming nothing more than the existence of some agents who are natural borrowers and others who are natural intermediaries or lenders. Additionally, the model with heterogeneous beliefs is extremely tractable and allows for extensive theoretical analysis. Even though a model with more realistic institutional frictions could provide additional insight, such a model would likely be more difficult to solve.

7 Conclusion

How are assets' payoffs allocated in markets with reusable collateral? In an environment where agents are ordered by their optimism, chains of lending emerge endogenously. As a consequence of belief heterogeneity, payoffs are tranched: agents begin to take losses only when their direct debtors are wiped out. Belief heterogeneity also leads to intermediation. Types with intermediate optimism act as intermediaries by borrowing from pessimistic agents and lending to optimists. These chains of collateralized lending resemble those that arise in financial markets through contracts such as CDOs and CLOs.

Intermediation creates heterogeneity in risk profiles and allows agents to take positions that match their beliefs. When an intermediary enters, more optimistic borrowers are able to take greater risks while more pessimistic lenders make safer loans. Intermediation is critical in bridging the gap between agents who want to take risky loans and the pessimistic agents who hold the majority of the wealth.

In extensions of my model, I study several applications of empirical relevance. This framework sheds light on the fact that increases in margins may be contagious both across markets and along the chain of lending, both of which are documented empirical phenomena. The observation that levered lenders as well as borrowers may be wealth-constrained allows my model to capture these phenomena in a parsimonious way. Second, endogenous intermediation produces an environment where leverage, intermediation capacity, and the quantity of safe assets all move together. Finally, I show that financial innovation leads to greater risk-taking either on behalf of end-borrowers or intermediaries who lend to those borrowers, both of which can have serious consequences for asset markets after bad news arrives.

A challenge for future empirical work is to disentangle the causes of crises in collateralized debt markets. In my model, the source of essentially all crises in these markets are increases in pessimism that cause intermediaries to become wealth-constrained and cut back on risky lending. In other theories, haircuts rise because of asymmetric information (such as in Dang, Gorton, and Hölmstrom (2013)) or because volatility spooks cash lenders such as money market funds (as in Brunnermeier and Pedersen (2009)).

Appendix

Proof of Proposition 2.1:

Proof. Observe that $A^1 = \{\min\{\theta, s\} : \theta \in S\}$. The payoff of an asset in A^2 is then $\min\{\theta_1, \min\{\theta_2, s\}\} = \{\min\{\theta_1, \theta_2\}, s\}$, but this asset is in A^1. $\square$

Proof of claim in Section 2.1:

Proof. Suppose first that the constraint $a + \int_{\theta \geq \hat{\theta}} d\mu_+(\theta) \geq \int_{\theta \geq \hat{\theta}} d\mu_-(\theta)$ is satisfied for all $\hat{\theta}$. Define $h(\hat{\theta}) = \inf_{\theta}(a + \int_{\theta' \geq \theta} d\mu_+(\theta') \leq \int_{\theta' \geq \hat{\theta}} d\mu_-(\theta'))$. Then observe that whenever $\mu_-([\theta_1, \theta_2]) > 0$, it must be that $\mu_+([h(\theta_1), h(\theta_2)]) > 0$ as well. Equivalently, defining a new measure $\tilde{\mu}_+([\theta_1, \theta_2]) = \mu_+([h(\theta_1), h(\theta_2)])$, μ_- is absolutely continuous with respect to $\tilde{\mu}_+$. By the Radon-Nikodym theorem, there exists a function $\beta(\theta)$ such that $\int_U d\mu_-(\theta) = \int_U \beta(\theta) d\tilde{\mu}_+(\theta)$ for any measurable U. Let $\alpha(\theta, \theta') = \mathbf{1}\{\theta = h(\theta')\}\beta(\theta')$. Then

$$\int_{\theta, \theta' \in [\theta_1, \theta_2]} \alpha(\theta, \theta') d\mu_+(\theta) = \int_{\theta' \in [\theta_1, \theta_2]} \beta(\theta') d\mu_+(h(\theta')) = \int_{\theta' \in [\theta_1, \theta_2]} \beta(\theta') d\tilde{\mu}_+(\theta') = \mu_-([\theta_1, \theta_2])$$

Define $\mu(U) = \int_U \alpha(\theta, \theta') d\mu_+(\theta)$. Then $\mu \geq 0$, as in the second optimization problem.

Now suppose $\mu \geq 0$, and let $\mu_+(U) = \int_{\theta \in U, \theta'} d\mu(\theta, \theta')$, $\mu_-(U) = \int_{\theta \in U, \theta'} d\mu(\theta', \theta)$. Then

$$\int_{\theta \geq \hat{\theta}} d\mu_+(\theta) = \int_{\theta \geq \hat{\theta}, \theta'} d\mu(\theta, \theta') \geq \int_{\theta, \theta' \geq \hat{\theta}} d\mu(\theta, \theta') = \int_{\theta \geq \hat{\theta}} d\mu_-(\theta)$$

using the fact that μ puts zero weight on sets $[\theta_1, \theta_2] \times [\theta_1', \theta_2']$ with $\theta_1' > \theta_2]$. $\square$

Proof of Lemma 3.1:

Proof. Define

$$f_{nm}(\theta, \theta') = \lambda_n E_n[\min\{\theta, s\} - \min\{\theta', s\}] - \lambda_m E_m[\min\{\theta, s\} - \min\{\theta', s\}]$$

Observe that $f_{nm}(\theta, \theta) = 0$ for all θ. The derivatives of f_{nm} are

$$f_{nm,1}(\theta, \theta') = \lambda_n(1 - F_n(\theta)) - \lambda_m(1 - F_m(\theta))$$

$$f_{nm,2}(\theta, \theta') = \lambda_m(1 - F_m(\theta')) - \lambda_n(1 - F_n(\theta'))$$

The hazard rate inequality implies $\frac{1-F_n(\theta)}{1-F_m(\theta)}$ is an increasing function of θ, so if $f_{nm,1}(0,0) < 0$ and $f_{nm,1}(\bar{s}, \bar{s}) > 0$, there exists θ^*_{nm} such that

- $\frac{1-F_n(\theta^*_{nm})}{1-F_m(\theta^*_{nm})} = \frac{\lambda_m}{\lambda_n}$,
- $f_{nm,1}(\theta, \theta') > 0$ iff $\theta > \theta^*_{nm}$,
- $f_{nm,2}(\theta, \theta') > 0$ iff $\theta < \theta^*_{nm}$.

For $\theta < \theta^*_{nm}$, define $\overline{g}_{nm}(\theta)$ to be the $\hat{\theta}$ that satisfies

$$\int_{\theta^*_{nm}}^{\hat{\theta}} f_{nm,1}(\theta', \theta) d\theta' = -\int_{\theta}^{\theta^*_{nm}} f_{nm,1}(\theta', \theta) d\theta'$$

if it exists and $\overline{g}_{nm}(\theta) = \bar{s}$ otherwise. Note that the integral on the left-hand side is positive and the one on the right is negative. The value $\hat{\theta} = \overline{g}_{nm}(\theta)$ is weakly decreasing in θ because the integral on the right-hand side is increasing in θ. By the fundamental theorem of calculus, $f_{nm}(\hat{\theta}, \theta) = 0$, and for $\theta' > \overline{g}_{nm}(\theta)$, $f_{nm}(\theta', \theta) > 0$ (because the derivative $f_{nm,1}$ is positive to the right of θ^*_{nm}). Hence statement (1) in the lemma has been proved. Statement (2) follows in an analogous fashion.

Statement (3) in the lemma follows from the observation that $f_{nm}(\theta, \theta) = 0$ and $f_{nm,1}(\theta', \theta) > 0$ for all $\theta' \geq \theta$ implies $f_{nm}(\theta', \theta) > 0$ for all $\theta' \geq \theta$. Given that $f_{nm,2} > 0$ for $\theta' < \theta^*_{nm}$, it is always the case that $f_{nm}(\theta', \theta) < 0$ for $\theta < \theta'$. $\square$

Proof of Proposition 3.2:

Proof. Let $\{r_n\}_{n=1}^N$ be the equilibrium values of

$$\max\left\{1, \max_{(\theta,\theta')} \frac{E_n[\min\{\theta, s\} - \min\{\theta', s\}]}{q(\theta) - q(\theta')}\right\}$$

and set $\lambda_n = \frac{1}{r_n}$. Then for each (θ, θ'),

$$q(\theta) - q(\theta') \geq \max_m \lambda_m E_m[\min\{\theta, s\} - \min\{\theta', s\}]$$

Define $\{\theta^*_n\}_{n=1}^N, \{\Theta^*_n\}_{n=1}^N$ in the same way as in Section 3.1, and let

$$\Theta^{opt}_n = \{(\theta, \theta') : q(\theta) - q(\theta') = \lambda_n E_n[\min\{\theta, s\} - \min\{\theta', s\}]\}$$

Note that the results of Section 3.1 imply $\Theta_n^{opt} \subset \Theta_n^*$. Optimality implies $\mu_n(\hat{A}\backslash\Theta_n^{opt}) = 0$ for all n, so it must be that $\mu_n(\hat{A}\backslash\Theta_n^*) = 0$. Then for any agent n, given $s < \theta_n^*$,

$$\int(\min\{\theta, s\} - \min\{\theta', s\})d\mu_n(\theta, \theta') = \int(\min\{\theta, s\} - \min\{\theta', s\})\mathbf{1}(\theta' < \theta_n^*)d\mu_n(\theta, \theta') = 0$$

Furthermore, for any $s > \theta_{n-1}^*$,

$$\int(\min\{\theta, s\} - \min\{\theta', s\})d\mu_n(\theta, \theta') = \int(\theta - \theta')d\mu_n(\theta, \theta') \equiv c_n$$

That is, if $s > \theta_{n-1}^*$, agent n's payoff is equal to a constant that is independent of s. Agent n bears risk only in the region $\theta_n^* \leq s \leq \theta_{n-1}^*$.

Agent n's payoff is 0 for $s \leq \theta_n^*$, some linear function $a_n s + b_n$ for $\theta_n^* \leq s \leq \theta_{n-1}^*$, and a constant c_n for $s \geq \theta_{n-1}^*$. The conditions that the payoff should be piecewise linear and continuous come from the fact that the payoffs of all assets (θ, θ') are piecewise linear and continuous. Continuity implies $b_n = -a_n\theta_n^*$ and $c_n = a_n(\theta_{n-1}^* - \theta_n^*)$. Market clearing will pin down the constant a_n.

For $\theta_n^* < s < \theta_{n-1}^*$, the market clearing condition is

$$\begin{aligned} s &= \sum_{m=1}^{N} \int(\min\{\theta, s\} - \min\{\theta', s\})d\mu_m(\theta, \theta') \\ &= a_n(s - \theta_n^*) + \sum_{m>n} a_m(\theta_{m-1}^* - \theta_m^*) \\ &= a_n s + C \end{aligned}$$

Equating coefficients on the left- and right-hand sides, it immediately follows that $a_n = 1$ for all n. Thus agent n's payoff is zero for $s \leq \theta_n^*$, $s - \theta_n^*$ for $\theta_n^* \leq s \leq \theta_{n-1}^*$, and $\theta_{n-1}^* - \theta_n^*$ for $s \geq \theta_{n-1}^*$, just as in the social planner's problem.

In this setting, the law of one price holds: assets with equal payoffs have equal prices. The total price agent n pays to acquire assets is then

$$q(\theta_{n-1}^*) - q(\theta_n^*) = \lambda_n E_n[\min\{\theta_n^*, s\} - \min\{\theta_{n-1}^*, s\}]$$

If there is $[\theta', \theta] \subset [\theta_n^*, \theta_{n-1}^*]$ such that

$$q(\theta) - q(\theta') > \lambda_n E_n[\min\{\theta, s\} - \min\{\theta', s\}]$$

it follows that

$$\begin{aligned}
q(\theta^*_{n-1}) - q(\theta^*_n) &= q(\theta^*_{n-1}) - q(\theta) + q(\theta) - q(\theta') + q(\theta') - q(\theta^*_n) \\
&> \lambda_n \Big(E_n[\min\{\theta^*_n, s\} - \min\{\theta, s\}] + E_n[\min\{\theta, s\} - \min\{\theta', s\}] \\
&+ E_n[\min\{\theta', s\} - \min\{\theta^*_{n-1}, s\}] \Big) \\
&= \lambda_n E_n[\min\{\theta^*_n, s\} - \min\{\theta^*_{n-1}, s\}]
\end{aligned}$$

yielding a contradiction. Hence $q(\theta) - q(\theta') = \lambda_n E_n[\min\{\theta, s\} - \min\{\theta', s\}]$ for all $[\theta', \theta] \subset [\theta^*_n, \theta^*_{n-1}]$, as desired. □

Proof of Proposition 3.3: Existence

Proof. The proof proceeds by an application of Brouwer's Fixed Point Theorem. Let $S = \{(\theta_1, \ldots, \theta_{N-1}) : 0 \leq \theta_{N-1} \leq \cdots \leq \theta_1 \leq \bar{s}\}$. Define a map $f : S \to S$ as follows:

1. Set
$$u_1(\theta_1) = E_1[\max\{s - \theta_1, 0\}]$$
$$u_n(\theta_{n-1}, \theta_n) = E_n[\min\{\theta_{n-1}, s\} - \min\{\theta_n, s\}],\ 1 < n < N$$
$$u_N(\theta_{N-1}) = E_N[\min\{\theta_{N-1}, s\}]$$

2. Define $\lambda_n = (\max\{\frac{u_n}{w_n}, 1\})^{-1}$ for $1 \leq n \leq N$.

3. Solve the problem
$$\max_{\bar{s} \geq \tilde{\theta}_1 \geq \cdots \geq \tilde{\theta}_N \geq 0} \lambda_1 E_1[\max\{s - \tilde{\theta}_1, 0\}] + \sum_{n=2}^{N-1} \lambda_n E_m[\min\{\tilde{\theta}_{n-1}, s\} - \min\{\tilde{\theta}_n, s\}] + \lambda_N E_N[\min\{\theta_{N-1}, s\}]$$
in order to obtain the optimum $(\theta'_1, \ldots, \theta'_{N-1})$.

4. Define $f(\theta_1, \ldots, \theta_{N-1}) = (\theta'_1, \ldots, \theta'_{N-1})$.

The map f is continuous and the values $\theta_1, \ldots, \theta_{N-1}$ correspond to an equilibrium in the sense of Proposition 3.2, as will be shown below.

To see that f is continuous, note that f is a composition of the function g that maps $(\theta_1, \ldots, \theta_{N-1})$ to $(\lambda_1, \ldots, \lambda_N)$ with the function h that maps $(\lambda_1, \ldots, \lambda_N)$ to $(\theta'_1, \ldots, \theta'_{N-1})$. Note that g is itself a composition of a continuous function of $(\theta_1, \ldots, \theta_{N-1})$ and the max function, so g is continuous.

The proof that h is continuous is slightly more involved. For $n > m$, define θ^*_{nm} to be the unique solution to the equation

$$\frac{\lambda_m}{\lambda_n} = \frac{1 - F_n(\theta)}{1 - F_m(\theta)} \equiv \psi(\theta)$$

if such a value θ^*_{nm} exists. Otherwise, if $\frac{\lambda_m}{\lambda_n} > \psi(\theta)$ for all θ, set $\theta^*_{nm} = \overline{s}$. Finally, if $\frac{\lambda_m}{\lambda_n} < \psi(\theta)$ for all θ, set $\theta^*_{nm} = 0$. As long as $\psi(\theta)$ is continuous, this mapping from λ_n, λ_m to θ^*_{nm} is also continuous. As shown in Section 3.2, the solution $(\theta'_1, \ldots, \theta'_{N-1})$ can be expressed recursively as $\theta'_{N-1} = \theta^*_{N-1,N}$, $\theta'_n = \max\{\theta_{n-1}, \max\limits_{m<n} \theta^*_{nm}\}$. This defines a continuous function from $\{\theta^*_{nm}\}_{n>m}$ to $(\theta'_1, \ldots, \theta'_{N-1})$, so h is continuous as well, meaning f is continuous.

The function $f : S \to S$ is therefore a continuous function on a compact, convex set in $\mathbb{R}^{N-1}$, so f has a fixed point $\theta^* = (\theta^*_1, \ldots, \theta^*_{N-1})$ by Brouwer's Fixed Point Theorem. It remains to show that this fixed point corresponds to an equilibrium.

With some abuse of notation, let $\lambda_n = (\max\{\frac{u_n(\theta^*)}{w_n}, 1\})^{-1}$, and set cash holdings $c_n = w_n - \lambda_n u_n(\theta^*)$. Furthermore, let $\mu_n(\theta^*_{n-1}, \theta^*_n) = 1$ (with $\theta^*_0 = \overline{s}$, $\theta^*_N = 0$). Recall that

$$\lambda_n E_n[\min\{\theta, s\} - \min\{\theta', s\}] = \max_m \lambda_m E_m[\min\{\theta, s\} - \min\{\theta', s\}]$$

as long as $[\theta', \theta] \subset [\theta^*_n, \theta^*_{n-1}]$. Then set $q(\theta) - q(\theta') = \lambda_n E_n[\min\{\theta, s\} - \min\{\theta', s\}]$ for all such pairs (θ, θ'). Clearly, then, $q(\theta) - q(\theta') \geq \max\limits_m \lambda_m E_m[\min\{\theta, s\} - \min\{\theta', s\}]$, as required in equilibrium. As shown in Section 3.2, (μ_n, c_n) solves type n's optimization problem taking $q(\theta)$ as given, since for all $(\theta', \theta) \not\subset [\theta^*_n, \theta^*_{n-1}]$, $\frac{u_n(\theta,\theta')}{q(\theta)-q(\theta')} < \frac{1}{\lambda_n} = \frac{u_n(\theta^*_{n-1},\theta^*_n)}{q(\theta^*_{n-1})-q(\theta^*_n)}$ (as long as $\lambda_n > 1$). When $\lambda_n = 1$, the same result holds.

For the converse result, first fix some $\lambda = \{\lambda_n\}_{n=1}^N \in \Lambda$ and let $\Theta^*_n = \{\theta : \lambda_n(1 - F_n(\theta)) = \max\limits_m \lambda_m(1 - F_m(\theta))\}$. Then define $\mu_n(\theta, \theta') = \mathbf{1}\{\theta = \sum \theta^*_n, \theta' = \inf \Theta^*_n\}$. This allocation is identical to the one obtained from the social planner's problem with weights λ. The uniqueness of this mapping follows from the uniqueness of the solution to the planner's problem. □

Proof of Proposition 3.3: Uniqueness

Proof. Consider two equilibria (1 and 2) such that n_1 types invest all of their wealth in the underlying asset or debt contracts in equilibrium 1 and $n_2 \geq n_1$ types do so in equilibrium 2. Suppose that, as in Proposition 3.2, the cutoff values $\{\theta^i_j\}$ correspond to equilibrium i. That is, in equilibrium i, the price function $q_i(\theta)$ satisfies

$$q_i(\theta) - q_i(\theta') = \lambda_j E_j[\min\{\theta, s\} - \min\{\theta', s\}]$$

whenever $[\theta', \theta] \subset [\theta^i_j, \theta^i_{j-1}]$. Let k be the lowest integer such that $\theta^1_k \neq \theta^2_k$.

First consider the case in which $\theta^1_k < \theta^2_k$. Define

$$v^i_j = \prod_{l=1}^{n_i} \frac{1 - F_l(\theta^i_l)}{1 - F_{l+1}(\theta^i_l)}$$

The k-th budget constraint in equilibrium i is

$$w_k = \frac{1}{v^i_{k+1}} \frac{1 - F_{k+1}(\theta^i_k)}{1 - F_k(\theta^i_k)} E_k[\min\{\theta^i_{k-1}, s\} - \min\{\theta^2_k, i\}]$$

Since $\theta^1_k < \theta^2_k$, $\theta^1_{k-1} = \theta^2_{k-1}$, it must be that $v^1_{k+1} > v^2_{k+1}$. Now assume $\theta^1_j < \theta^2_j$, $v^1_{j+1} > v^2_{j+1}$ for some $j > k$. Then

$$w_{j+1} = \frac{1}{v^i_{j+2}} \frac{1 - F_{j+2}(\theta^i_{j+1})}{1 - F_{j+1}(\theta^i_{j+1})} E_{j+1}[\min\{\theta^i_j, s\} - \min\{\theta^i_{j+1}, s\}]$$

This equation implies $\theta^1_{j+1} < \theta^2_{j+1}$, since the right-hand side is decreasing in v^i_{j+1} and increasing in θ^i_j. It also implies $v^1_{j+2} > v^2_{j+2}$ once we have $\theta^1_{j+1} < \theta^2_{j+1}$. By induction, then, $v^1_{n_1+1} > v^2_{n_1+1}$, but this is impossible because $v^1_{n_1+1} = 1$ (since in equilibrium 1, type $n + 1$ holds cash).

Now suppose $\theta^1_k > \theta^2_k$. By the same line of reasoning as above, $v^1_{n_1+1} < v^2_{n_1+1}$ and $\theta^1_{n_1+1} > \theta^2_{n_1+1}$. Given that type $n_1 + 1$ invests all wealth in the underlying asset or debt contracts in equilibrium 2 but not in equilibrium 1, it must be that

$$w_{n_1+1} = \frac{1}{v^2_{n_1+1}} E_{n_1+1}[\min\{\theta^2_{n_1}, s\} - \min\{\theta^2_{n_1+1}, s\}] > E_{n_1+1}[\min\{\theta^1_{n_1+1}, s\}]$$

Given that $\theta^2_{n_1+1} > 0$, this is clearly impossible if $v^1_{n_1+1} < v^2_{n_1+1}$ and $\theta^1_{n_1+1} > \theta^2_{n_1+1}$. Therefore $\theta^1_k > \theta^2_k$ is also impossible, so two distinct equilibria cannot exist.

I have omitted the proof of uniqueness in some edge cases, but those cases follow using exactly the same reasoning as in the above arguments. □

Proof of Proposition 4.2

Proof. Suppose that when endowments are $\{w_i\}_{i=1}^N$, the equilibrium default cutoffs are $\{\theta^*_i\}_{i=1}^{N-1}$ and the most pessimistic type who invests is K. Now assume that when the wealth of type $m > 1$ is $w'_m > w_m$, the new equilibrium default cutoffs are $\{\theta^{*'}_i\}_{i=1}^N$ and that $\theta^{*'}_1 < \theta^*_1$.

Note that in the new equilibrium, all agents $n < m$ must participate, and $\frac{w_1}{w'_n} \leq \frac{w_1}{w_n}$ for all n. Then, denoting $E_n[\min\{\theta, s\} - \min\{\theta', s\}]$ by $E_n(\theta, \theta')$,

$$\begin{aligned}\frac{w_1}{w_2} &= \frac{1 - F_2(\theta_1^{*'})}{1 - F_1(\theta_1^{*'})} \frac{E_1(\bar{s}, \theta_1^{*'})}{E_2(\theta_1^{*'}, \theta_2^{*'})} \\ &\geq \frac{1 - F_2(\theta_1^*)}{1 - F_1(\theta_1^*)} \frac{E_1(\bar{s}, \theta_1^*)}{E_2(\theta_1^*, \theta_2^{*'})}\end{aligned}$$

so it must be that $\theta_2^{*'} < \theta_2^*$. Assume by way of induction that $\theta_k^{*'} < \theta_k^*$ for all $k \leq n$, and observe that

$$\begin{aligned}\frac{w_1}{w_n'} &= \prod_{i=1}^{n} \frac{1 - F_i(\theta_{i-1}^{*'})}{1 - F_{i-1}(\theta_{i-1}^{*'})} \frac{E_1(\bar{s}, \theta_1^{*'})}{E_m(\theta_{n-1}^{*'}, \theta_n^{*'})} \\ &> \prod_{i=1}^{n} \frac{1 - F_i(\theta_{i-1}^*)}{1 - F_{i-1}(\theta_{i-1}^*)} \frac{E_1(\bar{s}, \theta_1^{*'})}{E_m(\theta_{n-1}^*, \theta_n^{*'})}\end{aligned}$$

where $\theta_n^{*'} \equiv 0$ if n holds only cash in the new equilibrium. On the other hand,

$$\frac{w_1}{w_n'} \leq \frac{w_1}{w_n} = \prod_{i=1}^{n} \frac{1 - F_i(\theta_{i-1}^*)}{1 - F_{i-1}(\theta_{i-1}^*)} \frac{E_1(\bar{s}, \theta_1^{*'})}{E_m(\theta_{n-1}^*, \theta_n^*)}$$

so $\theta_n^{*'} < \theta_n^*$. This is true for all $n \leq K$, so

$$w_1 = \prod_{i=1}^{K} \frac{1 - F_i(\theta_{i-1}^{*'})}{1 - F_{i-1}(\theta_{i-1}^{*'})} E_1(\bar{s}, \theta_1^{*'}) < \prod_{i=1}^{K} \frac{1 - F_i(\theta_{i-1}^*)}{1 - F_{i-1}(\theta_{i-1}^*)} E_1(\bar{s}, \theta_1^*)$$

a contradiction. Thus $\theta_1^{*'} \geq \theta_1^*$. The same argument by contradiction then applies if it is assumed that $\theta_2^{*'} < \theta_2^*$, and so on up to $\theta_{m-1}^{*'}$.

Now suppose $\theta_m^{*'} > \theta_m^*$. Given that $w_m' > w_m$, $\frac{w_m'}{w_n} > \frac{w_m}{w_n}$ for all $n > m$. Then

$$\begin{aligned}\frac{w_1}{w_{m+1}} &= \prod_{i=1}^{m+1} \frac{1 - F_i(\theta_{i-1}^{*'})}{1 - F_{i-1}(\theta_{i-1}^{*'})} \frac{E_1(\bar{s}, \theta_1^{*'})}{E_{m+1}(\theta_m^{*'}, \theta_{m+1}^{*'})} \\ &> \prod_{i=1}^{m+1} \frac{1 - F_i(\theta_{i-1}^*)}{1 - F_{i-1}(\theta_{i-1}^*)} \frac{E_1(\bar{s}, \theta_1^{*'})}{E_{m+1}(\theta_m^*, \theta_{m+1}^{*'})}\end{aligned}$$

but

$$\frac{w_1}{w_{m+1}} = \prod_{i=1}^{m+1} \frac{1 - F_i(\theta_{i-1}^*)}{1 - F_{i-1}(\theta_{i-1}^*)} \frac{E_1(\bar{s}, \theta_1^{*'})}{E_{m+1}(\theta_m^*, \theta_{m+1}^*)}$$

so it must be that $\theta_{m+1}^{*'} > \theta_{m+1}^*$. By induction, it is possible to show in exactly the same way that $\theta_n^{*'} > \theta_n^*$ for all $n > m$. Again, this yields a contradiction because

$$w_1 = \prod_{i=1}^{K} \frac{1 - F_i(\theta_{i-1}^{*'})}{1 - F_{i-1}(\theta_{i-1}^{*'})} E_1(\bar{s}, \theta_1^{*'}) < \prod_{i=1}^{K} \frac{1 - F_i(\theta_{i-1}^{*})}{1 - F_{i-1}(\theta_{i-1}^{*})} E_1(\bar{s}, \theta_1^{*}) = w_1$$

so it must be that $\theta_{m+1}^{*'} \leq \theta_{m+1}^*$. This argument can then be repeated to show that $\theta_{m+2}^{*'} \leq \theta_{m+2}^*$ once it is known that $\theta_{m+1}^{*'} \leq \theta_{m+1}^*$ and so forth until concluding that $\theta_n^{*'} \leq \theta_n^*$ for all $n > m$.

□

Proof of Proposition 4.4

Proof. Recall that $\theta_{i-1}^*, \theta_i^*$ satisfy $\frac{r_{i-1}}{r_i} = \frac{1-F_{i-1}(\theta_{i-1}^*)}{1-F_i(\theta_{i-1}^*)}, \frac{r_i}{r_{i+1}} = \frac{1-F_i(\theta_i^*)}{1-F_{i+1}(\theta_i^*)}$. The proposition then follows immediately from the observation that $\frac{1-F_j(s)}{1-F_k(s)}$ is increasing in s for $j < k$ (which, in turn, follows from Assumption 1). □

Proof of Proposition 5.1

Proof. First, observe that given θ_{N-1}^*, it is possible to derive the default thresholds θ_n^* of all other agents using the equilibrium conditions

$$w_n = \left(\prod_{k=n}^{N-1} \frac{1 - F_{k+1}(\theta_k^*)}{1 - F_k(\theta_k^*)} \right) E_n[\min\{\theta_{n-1}^*, s\} - \min\{\theta_n^*, s\}].$$

Clearly, given θ_k^* for $k > n$, it is possible to derive θ_n^* from this equation. The other boundary condition is that $\theta_0^* = \bar{s}$. Hence, finding an equilibrium reduces to finding a θ_{N-1}^* that yields $\theta_0^* = \bar{s}$.

Second, observe that for agent $N-1$, the equilibrium condition is

$$w_{N-1} = \frac{1 - F_N(\theta_{N-1}^*)}{1 - F_{N-1}(\theta_{N-1}^*)} E_{N-1}[\min\{\theta_{N-2}^*, s\} - \min\{\theta_{N-1}^*, s\}].$$

If $\tilde{F}_N(s) \geq F_N(s)$ for all s, then given a value of θ_{N-1}^*, the value of θ_{N-2}^* implied by this equation must be larger. Then, a short inductive argument implies that for any n, a larger value of θ_{N-1}^* implies a larger value of θ_n^*. Hence, once beliefs are changed from F_N to $\tilde{F}_N \geq F_N$, it must be that the equilibrium value of θ_{N-1}^* is lower than the original value in order to ensure $\theta_0^* = \bar{s}$.

Now note that the expression for margins can be rewritten as

$$m_n = \frac{w_n}{E_N[\min\{\theta^*_{N-1}, s\}] + \sum_{k=N}^{n+1} w_k}$$

since

$$w_n = \frac{1}{r_n} E_n[\min\{\theta^*_{n-1}, s\} - \min\{\theta^*_n, s\}]$$

for $n < N$. We have shown that the equilibrium value of θ^*_{N-1} decreases after beliefs shift to $\tilde{F}_N$, and endowments w_n are exogenous, so in fact $\tilde{m}_n \geq m_n$ for all n, as desired. □

Proof of Proposition 5.2

Proof. Let θ^*_{nA} be the riskiness of the debt contract taken by type $n \in \{2, 1A\}$ in the market for asset A, and define θ^*_{nB} analogously. The equilibrium conditions in this economy are

$$w_2 = \frac{1 - F_3(\theta^*_{2A})}{1 - F_2(\theta^*_{2A})} E_2[\min\{\theta^*_{1A}, s\} - \min\{\theta^*_{2A}, s\}] + \frac{1 - F_3(\theta^*_{2B})}{1 - F_2(\theta^*_{2B})} E_2[\min\{\theta^*_{1A}, s\} - \min\{\theta^*_{2A}, s\}],$$

$$w_{1A} = \frac{1 - F_3(\theta^*_{2A})}{1 - F_2(\theta^*_{2A})} \frac{1 - F_2(\theta^*_{1A})}{1 - F_1(\theta^*_{1A})} E_1[s - \min\{\theta^*_{1A}, s\}],$$

$$w_{1B} = \frac{1 - F_3(\theta^*_{2B})}{1 - F_2(\theta^*_{2B})} \frac{1 - F_2(\theta^*_{1B})}{1 - F_1(\theta^*_{1B})} E_1[s - \min\{\theta^*_{1B}, s\}],$$

and

$$\frac{1 - F_3(\theta^*_{2A})}{1 - F_2(\theta^*_{2A})} = \frac{1 - F_3(\theta^*_{2B})}{1 - F_2(\theta^*_{2B})}.$$

This final condition is key in the proof of the proposition– it equates the intermediary's returns across the different markets. Note that a choice of θ^*_{2A} pins down θ^*_{1A} through the budget constraint of type $1A$, but then this indifference condition pins down θ^*_{2B} as well (and thus θ^*_{1B}). Hence, a choice of θ^*_{2A} pins down the entire equilibrium. Importantly, a lower θ^*_{2A} implies a higher θ^*_{1A}, a higher θ^*_{1B}, and a lower θ^*_{2B}.

If the beliefs of type 3 change such that $\tilde{F}_{3A} \geq F_{3A}$, the values of θ^*_{1A} and θ^*_{1B} implied by a particular choice of θ^*_{2A} decrease. The value of θ^*_{2B} increases. Thus, for a particular choice of θ^*_{2A}, the right-hand side of the budget constraint of type 2 is lower under this change of beliefs. Therefore, θ^*_{2A} (and thus θ^*_{2B}) must be lower in the new equilibrium. The same technique used in the previous proof can be used to show that this implies higher margins for both the intermediary type 2 as well as both end-borrowers, types $1A$ and $1B$. In short, this is because the prices of both assets decrease, so the wealth of each end-borrower comprises a larger share of the corresponding asset's price. □

Proof of Proposition 5.3

Proof. First note that the condition

$$r_1 = \frac{1 - F_1(\theta_1^*)}{1 - F_2(\theta_1^*)} \frac{1 - F_2(\theta_2^*)}{1 - F_3(\theta_2^*)}$$

immediately implies that the equilibrium value of θ_1^* is decreasing in the value of θ_2^*. Next, note that since r_1 must always be greater than r_2, the price of the asset must be decreasing in θ_1^* and θ_2^*.

When type 2 becomes wealthier, the budget constraint of type 2 implies that for a given value of θ_2^*, it must be that θ_1^* is larger. That is, the curve defined by type 2's budget constraint shifts. The curve defined by type 1's indifference condition does not shift, nor does the price function, so in equilibrium it must be that θ_2^* decreases and θ_1^* increases. This implies that the most pessimistic agent makes a safer loan while the most optimistic agent takes riskier loans, as desired.

□

Proof of Proposition 5.5

Proof. Suppose that in the alternative economy, type 2 agents buy the asset directly. The equilibrium conditions are then

$$w_1 = \alpha \frac{1 - F_3(\theta_1^*)}{1 - F_1(\theta_1^*)} E_1[s - \min\{\theta_1^*, s\}],$$

$$w_2 = (1 - \alpha) \frac{1 - F_3(\theta_2^*)}{1 - F_2(\theta_2^*)} E_2[s - \min\{\theta_2^*, s\}],$$

where $\alpha \in (0, 1)$ represents the share of the asset purchased by type 1, the most optimistic agent.

If in the benchmark economy θ_1^* is smaller than in the alternative economy, $\theta_{1B}^* < \theta_{1A}^*$, then

$$\begin{aligned} w_1 &= \frac{1 - F_3(\theta_{2B}^*)}{1 - F_2(\theta_{2B}^*)} \frac{1 - F_2(\theta_{1B}^*)}{1 - F_1(\theta_{1B}^*)} E_1[s - \min\{\theta_{1B}^*, s\}] \\ &> \alpha \frac{1 - F_3(\theta_{1B}^*)}{1 - F_2(\theta_{1B}^*)} \frac{1 - F_2(\theta_{1B}^*)}{1 - F_1(\theta_{1B}^*)} E_1[s - \min\{\theta_{1B}^*, s\}] \\ &= \alpha \frac{1 - F_1(\theta_{1B}^*)}{1 - F_1(\theta_{1B}^*)} E_1[s - \min\{\theta_{1B}^*, s\}] \\ &> \alpha \frac{1 - F_3(\theta_{1A}^*)}{1 - F_1(\theta_{1A}^*)} E_1[s - \min\{\theta_{1A}^*, s\}] = w_1, \end{aligned}$$

which is a contradiction. This proves that $\theta^*_{1A} \leq \theta^*_{1B}$, so type 1 takes greater risk in the benchmark economy.

If in the alternative economy type 2 agents lend to type 1, then it is automatically true that they take riskier loans in the benchmark economy. In the alternative economy, they do not take loans at all.

$\square$